# The de Sitter Space-Time and the Riemannian Space $\mathbb{H}^4$: Two Four-Dimensional Surfaces of Hyperboloids Embedded within the Five-Dimensional Minkowski Space-Time

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## Contents

This manuscript was typed in $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX, the graphics were produced with GeoGebra.

# Introduction

The following pages contain my notes on the de Sitter space-time and the co-ordinate systems that can be used to describe it. The results are not new and have been published elsewhere, for example by
Y. Kim, C. Y. Oh, and N. Park: *Classical Geometry of de Sitter Spacetime: An Introductory Review*, `arXiv: hep-th/0212326` and: Journal of the Korean Physical Society, **42**, 573 (2003),
or
M. Spradlin, A. Strominger, and A. Volovich: *Les Houches Lectures on de Sitter Space*: `arXiv: hep-th/0110007`.
I would also like to mention
U. Moschella: *The de Sitter and Anti-de Sitter Sightseeing Tour*: Séminaire Poincaré **1**, 1 (2005), `http://www.bourbaphy.fr/moschella.pdf` .

However, the detailed calculations that lead to the final expressions of the metrical coefficients in the various co-ordinate systems are not given there.

It is an interesting feature intrinsic to the de Sitter space-time that there is no preferred co-ordinate system to stipulate how the four independent co-ordinates should be split up into time and space. The mathematical reason for this lies in the high symmetry: this space-time is of constant curvature. Physically speaking, it is not possible to distinguish time from space in a universe that is empty, with the exception of a vacuum energy. That is to say, if the universe were a de Sitter space-time, and it should have been of this kind during a potential inflationary phase, its aspect would depend upon the particular co-ordinates used by an imaginary observer: the time by which he set his clock, the length scale by which he calibrated his yardstick. According as which co-ordinate system has been chosen by the imaginary observer, the de Sitter universe might seem to him to expand inflationarily in different ways—or might appear to be static.

I have noted down the complete calculations in connection with the derivation of the respective metrical coefficients in the different co-ordinate systems. I do not lay claim that there might not be easier, more elegant, or more sophisticated ways of doing this.

Not only did I consider the de Sitter space-time $dS^4$, which can be regarded as a hyperboloid of one sheet immersed in the 5-dimensional Minkowski space-time $\mathbb{M}^5$, but also the respective hyperboloid of two sheets, $\mathbb{H}^4$, which is a Riemannian space with a definite metric, not a physical space-time. The following pages, doubtless, have a certain inclination towards a stroll in the garden of Riemannian geometry.

Finally, I would like to refer to Tolman's book,
R. C. Tolman: *Relativity, Thermodynamics, and Cosmology*, Dover Publications (Reprint), Chapter X,
as a source in which many properties of the de Sitter universe are extensively explained and discussed.

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# 1 Minkowski Space-Time

The five-dimensional Minkowski space-time $\mathbb{M}^5$ is equipped with the metric

$$\begin{aligned} ds^2 &= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 - (dx^4)^2 \\ &= \eta_{AB}\, dx^A\, dx^B \ , \end{aligned} \tag{1}$$

the five-dimensional metrical tensor being

$$(\eta_{AB}) = \mathrm{diag}(1, -1, -1, -1, -1) \ . \tag{2}$$

To avoid confusion, in what follows,

- capital Latin letters will run from 0 to 4,
- small Latin letters from 1 to 3,
- small Greek letters from 0 to 3,
- capital Greek letters from 1 to 4.

# 2 Subspaces embedded within $\mathbb{M}^5$

By imposing constraints upon the five co-ordinates, several subspaces within $\mathbb{M}^5$ can be specified. Examples are

1. the Hyperboloid

$$\mathbb{H}^4 = \left\{ x \in \mathbb{M}^5 \,\middle|\, \eta_{AB}\, x^A x^B = R^2 \right\} \quad ,$$

or

$$\mathbb{H}^4 = \left\{ x \in \mathbb{M}^5 \,\middle|\, (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2 = R^2 \right\} \quad , \tag{3}$$

2. the de Sitter space-time

$$dS^4 = \left\{ x \in \mathbb{M}^5 \,\middle|\, \eta_{AB}\, x^A x^B = -R^2 \right\} \quad ,$$

or

$$dS^4 = \left\{ x \in \mathbb{M}^5 \,\middle|\, (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2 = -R^2 \right\} \quad . \tag{4}$$

$\mathbb{H}^4$ is a Riemannian subspace with a definite metric,* whilst $dS^4$ is a space-time: one sign makes all the difference. The latter is explained in numerous texts and textbooks on cosmology, and, very much in detail, by Kim et al.† as well as Spradlin et al.‡

*There are two different points of view as regards the nature of a Riemannian space, which are to be found in the literature: According to Eisenhart (*Riemannian Geometry*) or Eddington (*The Mathematical Theory of Relativity*), any space or space-time in which an element of length is defined by $|g_{\mu\nu}\, dx^\mu\, dx^\nu|$ is a Riemannian space. Other authors, however, such as Weyl (*Raum, Zeit, Materie*) or Laugwitz (*Differentialgeometrie*), in addition require that the fundamental quadratic form $g_{\mu\nu}\, dx^\mu\, dx^\nu$ be definite.

†`arXiv: hep-th/0212326` and: Journal of the Korean Physical Society, **42**, 573 (2003).

‡*Les Houches Lectures on de Sitter Space*: `arXiv: hep-th/0110007`.

## 3 The Subspace $\mathbb{H}^4$

**Proposition 1.** $\mathbb{H}^4 \subset \mathbb{M}^5$ *is a Riemannian subspace with a purely spatial metric.*

*Proof.* I introduce new co-ordinates $t$, $\omega^\Gamma$ ($\Gamma = 1, \ldots, 4$) by

$$x^0 = \pm R \,\mathrm{ch}\, \frac{t}{R} \;, \quad x^\Gamma = R \,\mathrm{sh}\, \frac{t}{R}\, \omega^\Gamma \;, \tag{5}$$

and upon the $\omega^\Gamma$ impose the condition (surface $\mathbb{S}^3$ of a unit sphere in $\mathbb{R}^4$):

$$(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 + (\omega^4)^2 = 1 \;,$$

in order to have the $x^A$ ($A = 0, \ldots, 4$) satisfy equation (3). On observing Einstein's summation convention, this can be written as

$$\omega_\Gamma\, \omega^\Gamma = -1 \quad , \tag{6}$$

with $\omega_\Gamma = \eta_{\Gamma\Delta}\, \omega^\Delta$, because the metrical tensor is diagonal, its diagonal elements $\eta_{11}, \ldots, \eta_{44}$ being $-1$ throughout (eqn [2]).

From equation (6) it immediately follows that

$$\omega_\Gamma\, d\omega^\Gamma = 0 \;. \tag{7}$$

To avail myself of an expression of the line-element in the new co-ordinates, I need the differentials $dx^A$, which I obtain from eqn (5):—

$$\begin{aligned} dx^0 &= \pm\, \mathrm{sh}\, \frac{t}{R}\, dt \quad , \\ dx^\Gamma &= \mathrm{ch}\, \frac{t}{R}\, \omega^\Gamma\, dt + R\, \mathrm{sh}\, \frac{t}{R}\, d\omega^\Gamma \quad . \end{aligned} \tag{8}$$

With these, the line-element (1) takes the form

$$\begin{aligned} ds^2 &= (dx^0)^2 + \eta_{\Gamma\Delta}\, dx^\Gamma\, dx^\Delta \\ &= \mathrm{sh}^2\, \frac{t}{R}\, dt^2 - \sum_{\Gamma=1}^{4} \left( \mathrm{ch}\, \frac{t}{R}\, \omega^\Gamma\, dt + R\, \mathrm{sh}\, \frac{t}{R}\, d\omega^\Gamma \right)^2 \\ &= \mathrm{sh}^2\, \frac{t}{R}\, dt^2 \; + \\ &\quad + \mathrm{ch}^2\, \frac{t}{R}\, \omega_\Gamma\, \omega^\Gamma\, dt^2 + 2R\, \mathrm{sh}\, \frac{t}{R}\, \mathrm{ch}\, \frac{t}{R}\, dt\, \omega_\Gamma\, d\omega^\Gamma + R^2\, \mathrm{sh}^2\, \frac{t}{R}\, d\omega_\Gamma\, d\omega^\Gamma \\ &\overset{(6),(7)}{=} -dt^2 - R^2\, \mathrm{sh}^2\, \frac{t}{R}\, \delta_{\Gamma\Delta}\, d\omega^\Gamma\, d\omega^\Delta\, |_{\mathbb{H}^4} \quad . \end{aligned} \tag{9}$$

All distances, therefore, are space-like, and so are the co-ordinates. That is to say $\mathbb{H}^4$ is a Riemannian space with a space-like metric. As the r. h. s. of eqn (9) is a contraction of tensors, this is likewise true in any other co-ordinate system. □

The restriction "$|_{\mathbb{H}^4}$" in eqn (9) implies that of the four co-ordinates $\omega^\Gamma$ only three are independent. If I call these $\bar{x}^1$, $\bar{x}^2$, and $\bar{x}^3$, the line element in eqn (9) takes the form

$$ds^2 = -dt^2 - R^2 \operatorname{sh}^2 \frac{t}{R}\, \delta_{\Gamma\Delta} \frac{\partial\omega^\Gamma}{\partial\bar{x}^i} \frac{\partial\omega^\Delta}{\partial\bar{x}^j}\, d\bar{x}^i\, d\bar{x}^j \quad (i, j = 1, \ldots, 3) , \tag{10}$$

so defining the spatial metrical coefficients

$$a_{ij} := \delta_{\Gamma\Delta} \frac{\partial\omega^\Gamma}{\partial\bar{x}^i} \frac{\partial\omega^\Delta}{\partial\bar{x}^j} . \tag{11}$$

The metrical tensor of the metric induced on $\mathbb{H}^4$ by the constraint expressed in eqn (3) possesses four, not five, dimensions.

# 4 Spherical Co-Ordinates

## 4.1 $\mathbb{H}^4$

From the first eqn (5) it follows that $x^0 \in ]-\infty, -R] \cup [+R, \infty[$, which means there is a gap between $-R$ and $R$ in the co-ordinate $x^0$. But this is to be expected from a hyperboloid of two sheets. In order to choose spherical co-ordinates in which to describe $\mathbb{H}^4$, I specify the $\omega^\Gamma$ in eqn (5):—

$$\begin{aligned} \omega^1 &= \cos\chi , \\ \omega^2 &= \sin\chi \cos\vartheta , \\ \omega^3 &= \sin\chi \sin\vartheta \cos\varphi , \\ \omega^4 &= \sin\chi \sin\vartheta \sin\varphi . \end{aligned} \tag{12}$$

The condition $\eta_{\Gamma\Delta}\, \omega^\Gamma\, \omega^\Delta = -1$ is not upset by the introduction of these new co-ordinates. The manifold of all the points $\left(\omega^1 \,|\, \omega^2 \,|\, \omega^3 \,|\, \omega^4\right)$, therefore, is the surface $\mathbb{S}^3$ of a sphere immersed in $\mathbb{R}^4$.

The line element in eqn (9) could now be expressed by the differentials $d\chi$, $d\vartheta$, and $d\varphi$ by means of an elementary calculation, that is by expressing the squares of the differentials $d\omega^\Gamma$ by means of $d\chi$, $d\vartheta$, and $d\varphi$, and summing them in a second step:—

$$\begin{aligned} (d\omega^4)^2 &= (\cos\chi \sin\vartheta \sin\varphi\, d\chi + \sin\chi \cos\vartheta \sin\varphi\, d\vartheta + \sin\chi \sin\vartheta \cos\varphi\, d\varphi)^2 \\ &= \quad \ldots \end{aligned}$$

and accordingly for $(d\omega^3)^2$, $(d\omega^2)^2$, and $(d\omega^1)^2$. This is a possible, however clumsy undertaking.

It means less effort to exploit the transformation properties of tensors and compute the coefficients $a_{ij}$ defined by eqn (11):—

$$a_{ij} = \delta_{\Gamma\Delta} \frac{\partial\omega^\Gamma}{\partial\bar{x}^i} \frac{\partial\omega^\Delta}{\partial\bar{x}^j} = \sum_{\Gamma=1}^{4} \frac{\partial\omega^\Gamma}{\partial\bar{x}^i} \frac{\partial\omega^\Gamma}{\partial\bar{x}^j} \quad ,$$

where $(\bar{x}^i) = (\chi, \vartheta, \varphi)^T$. So $i, j = 1, 2, 3$, whilst $\Gamma, \Delta = 1, 2, 3, 4$.

For the coefficients $a_{ij}$ I find:—

$$\begin{aligned}
a_{11} &= \sum_{\Gamma=1}^{4} \left(\frac{\partial\omega^\Gamma}{\partial\chi}\right)^2 = 1 \quad , \\
a_{12} &= \sum_{\Gamma=2}^{4} \frac{\partial\omega^\Gamma}{\partial\chi} \frac{\partial\omega^\Gamma}{\partial\vartheta} \\
&= \sin\chi\cos\chi \Big(-\cos\vartheta\sin\vartheta + \sin\vartheta\cos\vartheta\cos^2\varphi \ + \\
&\quad + \ \sin\vartheta\cos\vartheta\sin^2\varphi\Big) = 0 \quad , \\
a_{13} &= \sum_{\Gamma=3}^{4} \frac{\partial\omega^\Gamma}{\partial\chi} \frac{\partial\omega^\Gamma}{\partial\varphi} \\
&= \sin\chi\cos\chi \left(-\sin^2\vartheta\sin\varphi\cos\varphi + \sin^2\vartheta\sin\varphi\cos\varphi\right) = 0 \quad , \\
a_{22} &= \sum_{\Gamma=2}^{4} \left(\frac{\partial\omega^\Gamma}{\partial\vartheta}\right)^2 \\
&= \sin^2\chi \left\{\sin^2\vartheta + \cos^2\vartheta\left(\cos^2\varphi + \sin^2\varphi\right)\right\} = \sin^2\chi \quad , \\
a_{23} &= \sum_{\Gamma=3}^{4} \frac{\partial\omega^\Gamma}{\partial\vartheta} \frac{\partial\omega^\Gamma}{\partial\varphi} \\
&= \sin^2\chi\,(-\sin\vartheta\cos\vartheta\sin\varphi\cos\varphi + \sin\vartheta\cos\vartheta\sin\varphi\cos\varphi) = 0 \quad , \\
a_{33} &= \sum_{\Gamma=3}^{4} \left(\frac{\partial\omega^\Gamma}{\partial\varphi}\right)^2 = \sin^2\chi\sin^2\vartheta \quad .
\end{aligned} \tag{13}$$

So the spatial metrical coefficients $a_{ij}$ are:

$$(a_{ij}) = \mathrm{diag}\left(1, \sin^2\chi, \sin^2\chi\sin^2\vartheta\right) \ . \tag{14}$$

This is the metrical tensor on the three-dimensional surface $\mathbb{S}^3$ of a four-dimensional unit-hypersphere which is immersed in $\mathbb{R}^4$ (see Eddington, p. 156). This is not unexpected, because in eqn (12) the co-ordinates $\chi$, $\vartheta$, and $\varphi$ from the beginning were so chosen that they outline the surface $\mathbb{S}^3$ which is defined by the constraint $-\eta_{\Gamma\Delta}\,\omega^\Gamma\,\omega^\Delta = 1$.

The line element on $\mathbb{H}^4$ according to eqs (10), (11), and (14) is

$$\begin{aligned} ds^2 &= -dt^2 - R^2 \operatorname{sh}^2 \frac{t}{R} \left\{ a_{\chi\chi}\, d\chi^2 + a_{\vartheta\vartheta}\, d\vartheta^2 + a_{\varphi\varphi}\, d\varphi^2 \right\} \\ &= -dt^2 - R^2 \operatorname{sh}^2 \frac{t}{R} \left\{ d\chi^2 + \sin^2\chi \left( d\vartheta^2 + \sin^2\vartheta\, d\varphi^2 \right) \right\} \quad , \end{aligned}$$

or

$$ds^2 = -dt^2 - R^2 \operatorname{sh}^2 \frac{t}{R} \left( d\chi^2 + \sin^2\chi\, d\Omega^2 \right) \quad , \tag{15}$$

using $d\Omega^2 := d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$, the squared line element on $\mathbb{S}^2$, for short.

### 4.2 $dS^4$

In the case of the de Sitter space-time $dS^4$, co-ordinates similar to those that are introduced by the equations (5) are called *global co-ordinates*. As $dS^4$ is a hyperboloid of one sheet, the co-ordinates have to be chosen differently from $\mathbb{H}^4$, as the $x^A$ must fulfil eqn (4):

$$x^0 = R \operatorname{sh} \frac{ct}{R} \; , \quad x^\Gamma = R \operatorname{ch} \frac{ct}{R}\, \omega^\Gamma \; , \quad \Gamma = 1, \ldots, 4, \tag{16}$$

where the co-ordinates $\omega^\Gamma$ are again confined to the surface $\mathbb{S}^3$ of a four-dimensional sphere by the restriction

$$\sum_{\Gamma=1}^{4} (\omega^\Gamma)^2 = 1 \; .$$

The $x^A$ thus chosen obey eqn (4).

As $dS^4$ is a physical space-time and $t$ denotes a physical co-ordinate time, I regard $ct$, not $t$ itself, as the temporal co-ordinate, which implies that $t$ is measured in units of time. I did not do this in the case of the Riemannian space $\mathbb{H}^4$, as $t$, there, is only a formal, not a physical "time". So I assigned, there, the unit of length to $t$.

Just like in the case of $\mathbb{H}^4$, I express the $\omega^\mu$ by the independent spherical co-ordinates $\chi$, $\vartheta$, and $\varphi$, as defined by eqs (12). The line element then takes the form:—

**Proposition 2.** *Expressed in the spherical co-ordinates of eqs* (12)*, the line element in $dS^4$ is*

$$ds^2 = c^2\, dt^2 - R^2 \operatorname{ch}^2 \frac{ct}{R} \left( d\chi^2 + \sin^2\chi\, d\Omega^2 \right) \; , \tag{17}$$

*where $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$ is the squared line element on $\mathbb{S}^2$.*

This metric is indefinite, so $dS^4$ is a space-time. Embedded is the sub*space* $\mathbb{S}^3$, which means that a spatially closed universe is described. $ds$ is of the Robertson-Walker form with the scale factor $\mathsf{R}(t) = R \operatorname{ch} \frac{ct}{R}$.[§]

*Proof of the proposition.* Let $(\bar{x}^\mu) = (ct, \chi, \vartheta, \varphi)^T$, as defined by eqs (16) and (12). The metrical tensor $g_{\mu\nu}(t, \chi, \vartheta, \varphi)$ (I omit the bar above the symbol $g_{\mu\nu}$) then results from Minkowski's by means of

$$g_{\mu\nu} = \eta_{AB} \frac{\partial x^A}{\partial \bar{x}^\mu} \frac{\partial x^B}{\partial \bar{x}^\nu} , \quad (A, B = 0, \ldots, 4;\ \mu, \nu = 0, \ldots, 3).$$

$$\begin{aligned}
g_{00} &= \frac{1}{c^2} \left( \frac{\partial x^0}{\partial t} \right)^2 - \frac{1}{c^2} \sum_{\Gamma=1}^{4} \left( \frac{\partial x^\Gamma}{\partial t} \right)^2 \\
&= \operatorname{ch}^2 \frac{ct}{R} - \operatorname{sh}^2 \frac{ct}{R} \underbrace{\sum_{\Gamma=1}^{4} (\omega^\Gamma)^2}_{1} = 1 . \\
g_{0i} &= \frac{1}{c} \eta_{AB} \frac{\partial x^A}{\partial t} \frac{\partial x^B}{\partial \bar{x}^i} = -\frac{1}{c} \sum_{\Gamma=1}^{4} \frac{\partial x^\Gamma}{\partial t} \frac{\partial x^\Gamma}{\partial \bar{x}^i} \\
&= -R \operatorname{sh} \frac{ct}{R} \operatorname{ch} \frac{ct}{R} \underbrace{\sum_{\Gamma=1}^{4} \omega^\Gamma \frac{\partial \omega^\Gamma}{\partial \bar{x}^i}}_{\substack{=0, \\ \text{as } \sum_{\Gamma=1}^{4} (\omega^\Gamma)^2 = 1.}} = 0 . \\
g_{ij} &= \eta_{AB} \frac{\partial x^A}{\partial \bar{x}^i} \frac{\partial x^B}{\partial \bar{x}^j} = -R^2 \operatorname{ch}^2 \frac{ct}{R} \underbrace{\sum_{\Gamma=1}^{4} \frac{\partial \omega^\Gamma}{\partial \bar{x}^i} \frac{\partial \omega^\Gamma}{\partial \bar{x}^j}}_{=: a_{ij}} .
\end{aligned}$$

$a_{ij}$ is the same spatial metrical tensor as the one defined by eqn (11), because the $\omega^\Gamma(\bar{x}^i)$ are the same functions as those defined by eqs (12). Hence:

$$(a_{ij}) = \operatorname{diag}\left(1, \sin^2 \chi, \sin^2 \chi \sin^2 \vartheta\right) ,$$

and, with $d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2$, the square of the line element is

$$ds^2 = c^2 dt^2 - R^2 \operatorname{ch}^2 \frac{ct}{R} \left( d\chi^2 + \sin^2 \chi \, d\Omega^2 \right) . \qquad \square$$

The spatial part of this Robertson-Walker metric is one of the surface $\mathbb{S}^3$ of a sphere immersed in $\mathbb{R}^4$, as it is in the case of $\mathbb{H}^4$. The scale factor, the radius of this sphere, however, shows a behaviour different from what is found in $\mathbb{H}^4$:—

[§]I use the sanserif $\mathsf{R}$ to designate the scale factor because I used up the normal $R$ for the radius of the respective hyperboloid.

### 4.3 Comparison of the Temporal Behaviours of $dS^4$ and $\mathbb{H}^4$

The global co-ordinates of $dS^4$ (eqs [16]) correspond to a "big bounce" model of the universe: the radius of the sphere shrinks as long as $t < 0$, reaches its minimal value $R$ at $t = 0$, and expands again. In $\mathbb{H}^4$, however, the respective sphere attains the radius zero, if $t = 0$ (eqs [5]).

This difference originates from the fact that $\mathbb{H}^4$ and $dS^4$ belong to two different kinds of hyperboloid. In the case of $dS^4$, $x^0$ becomes zero at $t = 0$, whilst the $x^\Gamma$ are confined to the surface $\mathbb{S}^3$ of a sphere of the finite minimum radius $R$: they outline the minimal circumference of the hyperboloid (see fig. 1).

In the case of $\mathbb{H}^4$, however, $x^0 = \pm R$ at $t = 0$, whilst the $x^\Gamma$ all vanish. This means that the lowest point of the upper sheet or the highest point of the lower sheet of this hyperboloid has been reached (see fig. 2).

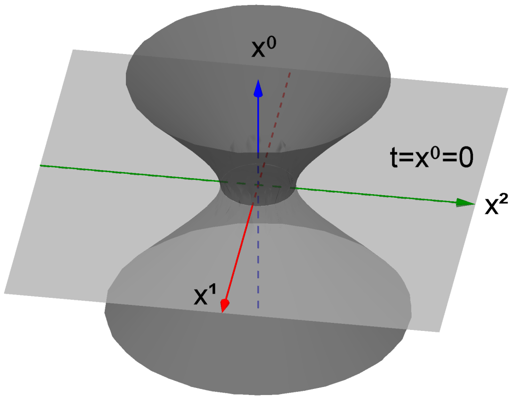


Figure 1: Hyperboloid of one sheet to illustrate the de Sitter space-time. At $t = 0$, $x^0 = 0$, and $x^1$ and $x^2$ are confined to the shortest circumference.

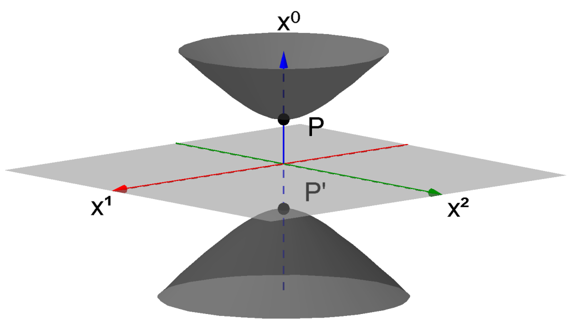


Figure 2: Hyperboloid of two sheets to illustrate the hyperbolic space $\mathbb{H}^4$. At $t = 0$, either the point $P$ or the point $P'$ is reached: $x^0$ is finite, whilst the $x^\Gamma$ all vanish.

## 4.4 Conformality with a Cylindrical Metric

### 4.4.1 $\mathbb{H}^4$

The metric in spherical co-ordinates of $\mathbb{H}^4$,

$$ds^2 = -dt^2 - R^2 \operatorname{sh}^2 \frac{t}{R} \left\{ d\chi^2 + \sin^2\chi \left( d\vartheta^2 + \sin^2\vartheta \, d\varphi^2 \right) \right\} \quad , \tag{15}$$

is conformal with that on a hypercylinder:—

**Proposition 3.** *The metric of eqn* (15) *is conformal with the "cylindrical" metric*

$$d\mathfrak{s}^2 = -dT^2 - R^2 \left\{ d\chi^2 + \sin^2\chi \left( d\vartheta^2 + \sin^2\vartheta \, d\varphi^2 \right) \right\} \quad , \tag{18}$$

*with the "conformal time" $T$. Between $ds$ and $d\mathfrak{s}$ the relation*

$$ds^2 = \operatorname{csch}^2 \frac{T}{R} \, d\mathfrak{s}^2 \tag{19}$$

*obtains.*

From the point of view of embedded spaces, the form of the "cylindrical" metric from eqn (18) implies that four of the five co-ordinates of the conformal Riemannian space are locked onto the surface $\mathbb{S}^3$ of a sphere in $\mathbb{R}^4$, whilst the the fifth, the "conformal time" $T$, may vary freely.

*Proof of the proposition.* Two metrics are conformal with each other, if there is a function that acts as a factor of proportionality between their respective line-elements, so that $ds^2 = \lambda \cdot d\mathfrak{s}^2$. To this end I rewrite the line-element of eqn (15) as

$$ds^2 = -\operatorname{sh}^2 \frac{t}{R} \left\{ \operatorname{csch}^2 \frac{t}{R} \, dt^2 + R^2 \left( d\chi^2 + \sin^2\chi \, d\Omega^2 \right) \right\} \quad , \tag{20}$$

with $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta \, d\varphi^2$, and introduce the "conformal time" $T$ by

$$dT = \operatorname{csch} \frac{t}{R} \, dt \quad . \tag{21}$$

Upon integrating, $T$ becomes

$$\frac{T}{R} = \ln \operatorname{th} \frac{t}{2R} \quad .$$

If I restrict $t$ to the interval $]0, \infty[$ , $T \in ] - \infty, 0[$ .

Upon inverting the ln, I obtain

$$e^{\frac{T}{R}} = \operatorname{th} \frac{t}{2R} \quad ,$$

and, on exploiting the formula

$$\operatorname{sh}\frac{t}{R} = 2\,\frac{\operatorname{th}\dfrac{t}{2R}}{1-\operatorname{th}^2\dfrac{t}{2R}} \quad ,$$

I can express $t$ by the "conformal time" $T$:—

$$\operatorname{sh}\frac{t}{R} = 2\,\frac{e^{\frac{T}{R}}}{1-e^{2\frac{T}{R}}} = -\operatorname{csch}\frac{T}{R} \quad . \tag{22}$$

Finally, eqs (21) and (22) are inserted into eqn (20):—

$$ds^2 = -\operatorname{csch}^2\frac{T}{R}\,\left\{dT^2 + R^2\left(d\chi^2 + \sin^2\chi\, d\Omega^2\right)\right\} \quad .$$

The term in braces is just $-d\mathfrak{s}^2$. Owing to the symmetry of the squared csch, the restriction $T \in\, ]-\infty, 0[$ can be lifted. $t = 0$, however, has to be exempted. □

### 4.4.2 $dS^4$

Like the metric of $\mathbb{H}^4$, the Robertson-Walker metric of $dS^4$, expressed in spherical co-ordinates,

$$ds^2 = c^2\,dt^2 - R^2\,\operatorname{ch}^2\frac{ct}{R}\left\{d\chi^2 + \sin^2\chi\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right\} , \tag{17}$$

is conformal with that of a "cylindrical" space-time of constant radius: a static Robertson-Walker metric with positive spatial curvature:—

**Proposition 4.** *The metric of eqn* (17) *is conformal with the static "cylindrical" metric*

$$d\mathfrak{s}^2 = dT^2 - R^2\left\{d\chi^2 + \sin^2\chi\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)\right\} \quad , \tag{23}$$

*with the conformal time $T$. Between $ds$ and $d\mathfrak{s}$ the relation*

$$ds = \sec\frac{T}{R}\,d\mathfrak{s}\ , \quad \frac{T}{R} \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[ , \tag{24}$$

*obtains.*

Compared with $\mathbb{H}^4$, only the sign of $dT^2$ is different, because this is a space-time. Secondly, the factor of proportionality between $ds$ and $d\mathfrak{s}$ is $\sec\frac{T}{R}$ in the present case, so confining $T/R$ to the interval $]-\pi/2, +\pi/2\,[$, which means that the conformal space-time is a cylinder of finite length. Apart from this, the conformal metrics are of the same static cylindrical form for both $\mathbb{H}^4$ and $dS^4$.

*Proof of the proposition.* Like in the proof of proposition 3 I extract the time-dependent function as a common factor from the r. h. s. of the expression for the squared line element (eqn [17]):—

$$ds^2 = \mathrm{ch}^2 \frac{ct}{R} \left\{ \mathrm{sech}^2 \frac{ct}{R} c^2 \, dt^2 - R^2 \left( d\chi^2 + \sin^2 \chi \, d\Omega^2 \right) \right\} \quad , \tag{25}$$

with $d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2$, and introduce the conformal time $T$ (measured in units of length) by

$$dT = \mathrm{sech} \frac{ct}{R} c \, dt \quad . \tag{26}$$

Upon integrating, $T$ becomes

$$\frac{T}{R} = \mathrm{arctg} \, \mathrm{sh} \frac{ct}{R} \quad .$$

This restricts $T/R$ to the interval $]-\pi/2, +\pi/2[$.

From the last formula I get

$$\mathrm{ch}^2 \frac{ct}{R} = 1 + \mathrm{sh}^2 \frac{ct}{R} = 1 + \mathrm{tg}^2 \frac{T}{R} = \sec^2 \frac{T}{R} \, ,$$

and from this as well as from eqs (25), (26):

$$ds^2 = \sec^2 \frac{T}{R} \left\{ dT^2 - R^2 \left( d\chi^2 + \sin^2 \chi \, d\Omega^2 \right) \right\} \, .$$

The term within the braces is just $d\mathfrak{s}^2$. □

## 5 Co-Ordinates in which to describe $dS^4$ and Particular Cosmological Models

The l. h. s. of Einstein's field equations, written in the form

$$R_{\mu\nu} - \tfrac{1}{2}\mathscr{R} g_{\mu\nu} + \Lambda g_{\mu\nu} = -\varkappa T_{\mu\nu} \, , \tag{27}$$

$\varkappa = 8\pi G/c^4$, $\mathscr{R} = g^{\varrho\sigma} R_{\varrho\sigma}$, depends solely upon the metrical coefficients of the underlying space-time, with $\Lambda$ being a free parameter. In the case of the Robertson-Walker metric,

$$ds^2 = c^2 \, dt^2 - \mathsf{R}^2(t) \left\{ d\chi^2 + f^2(\chi) \, d\Omega^2 \right\} \, , \tag{28}$$

$$f(\chi) = \begin{cases} \sin\chi & \text{(closed space)}, \\ \chi & \text{(Euclidean space)}, \\ \mathrm{sh}\,\chi & \text{(open space)}, \end{cases}$$

the nonvanishing components of the Ricci tensor of this metric are

$$\begin{aligned}
R_{00} &= \frac{3}{c^2}\frac{\ddot{\mathsf{R}}}{\mathsf{R}} \,, \\
R_{11} &= -\left(\frac{1}{c^2}\,\mathsf{R}\ddot{\mathsf{R}} + \frac{2}{c^2}\,\dot{\mathsf{R}}^2 + 2k\right) \,, \\
R_{22} &= -\left(\frac{1}{c^2}\,\mathsf{R}\ddot{\mathsf{R}} + \frac{2}{c^2}\,\dot{\mathsf{R}}^2 + 2k\right) f^2(\chi) \,, \\
R_{33} &= -\left(\frac{1}{c^2}\,\mathsf{R}\ddot{\mathsf{R}} + \frac{2}{c^2}\,\dot{\mathsf{R}}^2 + 2k\right) f^2(\chi)\,\sin^2\vartheta \,, \qquad (29) \\
k &= -\frac{f''(\chi)}{f(\chi)} = \begin{cases} 1 & \text{(closed space),} \\ 0 & \text{(Euclidean space),} \\ -1 & \text{(open space).} \end{cases}
\end{aligned}$$

This results in the curvature scalar

$$\begin{aligned}
\mathscr{R} &= g^{\varrho\sigma} R_{\varrho\sigma} \\
&= \frac{6}{c^2}\left(\frac{\ddot{\mathsf{R}}}{\mathsf{R}} + \frac{\dot{\mathsf{R}}^2}{\mathsf{R}^2} + \frac{kc^2}{\mathsf{R}^2}\right) \,,
\end{aligned}$$

with the metrical coefficients taken from eqn (28).

In de Sitter's empty world, the energy-momentum tensor $T_{\mu\nu}$ vanishes, so that, not only their left-hand sides, but the Einstein equations themselves depend solely upon the metric and, consequently, upon the co-ordinates chosen to describe $dS^4$. The 00 component of Einstein's equations results, if $T_{\mu\nu} = 0$, in the Friedmann equation

$$\dot{\mathsf{R}}^2 - \frac{\Lambda c^2}{3}\,\mathsf{R}^2 = -k\,c^2 \,. \qquad (30)$$

Now, the Robertson-Walker line element of $dS^4$ in spherical co-ordinates contains the scale factor

$$\mathsf{R}(t) = R\,\mathrm{ch}\,\frac{ct}{R} \,,$$

see Proposition 2, which gives

$$\dot{\mathsf{R}}^2 - \frac{c^2}{R^2}\,\mathsf{R}^2 = -c^2 \,.$$

But this coincides with the Friedmann equation (30), if $k = 1$ and $R^2 = 3/\Lambda$. That is to say a description of the geometrical properties of $dS^4$ in terms of global, or "spherical", co-ordinates makes it appear as a spatially closed universe, which evolves in a "big bounce" manner: $\mathsf{R}(t) \sim \mathrm{ch}(ct/R)$.

If $k = 0$ is assumed in eqn (30) (Euclidean space), the solution will be

$$\mathsf{R}(t) \sim e^{\sqrt{\frac{\Lambda}{3}}\, c\, t} \ , \tag{31}$$

whilst it is

$$\mathsf{R}(t) = \sqrt{\frac{3}{\Lambda}}\ \mathrm{sh}\left(\sqrt{\frac{\Lambda}{3}}\, c\, t\right) \ , \tag{32}$$

if $k = -1$ and, hence, a spatially open universe is assumed. It will be shown that the former case materialises, if $dS^4$ is described in planar co-ordinates (§ 6), and the latter, if hyperbolic co-ordinates are assumed (§ 7). The radius $R$ of the hyperboloid in $\mathbb{M}^5$ the surface of which is the de Sitter space-time determines the cosmological constant by the condition

$$R^2 = \frac{3}{\Lambda} \quad . \tag{33}$$

This shows that the particular way of splitting up the four independent co-ordinates in $dS^4$ into space and time not only determines the functional relationship between the Hubble "constant" $\dot{\mathsf{R}}(t)/\mathsf{R}(t)$ and the respective co-ordinate time, but also the spatial curvature, *as it appears in the chosen system of co-ordinates*. The co-ordinate time $t$ in spherical co-ordinates is different from that in flat, static, or hyperbolic co-ordinates.

# 6 Planar Co-Ordinates

The spherical or "global" co-ordinates do not pave the only way of splitting up the four independent co-ordinates on which $x^0, \ldots, x^4$ depend into one temporal and three spatial. The "planar co-ordinates" are an alternative co-ordinate system that separates time from space.

## 6.1 $\mathbb{H}^4$ in Planar Co-Ordinates

In the case of $\mathbb{H}^4$, the planar co-ordinates $t, y^i$ are introduced by

$$\begin{aligned} x^0 &= R\,\mathrm{ch}\,\frac{t}{R} - \frac{1}{2R}\,\eta_{ij}\, y^i\, y^j\, e^{\frac{t}{R}} \quad (i = 1,2,3) \quad , \\ x^i &= y^i\, e^{\frac{t}{R}} \quad , \\ x^4 &= R\,\mathrm{sh}\,\frac{t}{R} + \frac{1}{2R}\,\eta_{ij}\, y^i\, y^j\, e^{\frac{t}{R}} \quad . \end{aligned} \tag{34}$$

The metrical tensor of $\mathbb{M}^5$ is $(\eta_{AB}) = \mathrm{diag}(1, -1, -1, -1, -1)$.

**Proposition 5.** *Without further restrictions imposed on the $y^i$ (contrary to the $\omega^i$), the $x^A$ obey eqn* (3).

*Proof.* Throughout this proof, I use the notation $y_i = \eta_{ij}\, y^j$.

$$\begin{aligned}
x_A x^A &= \left(R \operatorname{ch} \frac{t}{R} - \frac{1}{2R}\, y_i\, y^i\, e^{\frac{t}{R}}\right)^2 + y_i\, y^i\, e^{2\frac{t}{R}} \; - \\
&\quad - \left(R \operatorname{sh} \frac{t}{R} + \frac{1}{2R}\, y_i\, y^i\, e^{\frac{t}{R}}\right)^2 \\
&= R^2 \operatorname{ch}^2 \frac{t}{R} - y_i\, y^i\, e^{\frac{t}{R}} \operatorname{ch} \frac{t}{R} + \frac{1}{4R^2}\, y_i\, y^i\, y_j\, y^j\, e^{2\frac{t}{R}} \; + \\
&\quad + y_i\, y^i\, e^{2\frac{t}{R}} \; - \\
&\quad - R^2 \operatorname{sh}^2 \frac{t}{R} - y_i\, y^i\, e^{\frac{t}{R}} \operatorname{sh} \frac{t}{R} - \frac{1}{4R^2}\, y_i\, y^i\, y_j\, y^j\, e^{2\frac{t}{R}} \\
&= R^2 + y_i\, y^i\, e^{\frac{t}{R}} \left\{ e^{\frac{t}{R}} - \left(\operatorname{sh} \frac{t}{R} + \operatorname{ch} \frac{t}{R}\right)\right\} \\
&= R^2 \quad .
\end{aligned}$$

□

The introduction of these co-ordinates means that the original constraint $\eta_{AB}\, x^A\, x^B = R^2$, as expressed in eqn (3), has been replaced by two:

1. In the $x^0 x^4$ plane: for given $y^i$ a hyperbola of radius

$$\sqrt{R^2 + \sum_{i=1}^{3} (y^i)^2\, e^{2\frac{t}{R}}}$$

   (real and imaginary axes of equal length) by

$$(x^0)^2 - (x^4)^2 = R^2 + \sum_{i=1}^{3} (y^i)^2\, e^{2\frac{t}{R}} \quad .$$

2. In the space $\mathbb{R}^3$ outlined by $x^1$, $x^2$, $x^3$: a rescaling of the radius vector $r = \delta_{ij}\, x^i\, x^j$ by

$$\sum_{i=1}^{3} (x^i)^2 = \sum_{i=1}^{3} (y^i)^2\, e^{2\frac{t}{R}} \quad .$$

In this way, the four independent co-ordinates are split up into "space" $(y^1, y^2, y^3)$ and "time" $t$ in a manner different from the spherical co-ordinates.

These co-ordinates cover only that part of $\mathbb{H}^4$ for which $x^0 + x^4 = R \exp(t/R) > 0$ obtains. This corresponds to the upper sheet of the hyperboloid, on which $x^0 > 0$, see fig. 3. The limiting plane, $x^0 + x^4 = 0$, corresponds to $t \to -\infty$. For finite

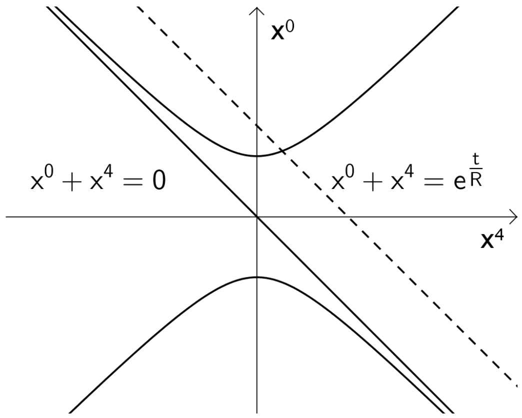


Figure 3: $\mathbb{H}^4$, illustrated by means of $\mathbb{H}^2$, and the dividing plane $x^0 + x^4 = 0$ ($\lim_{t\to-\infty}$). For any given finite value of $t$, the respective points on $\mathbb{H}^2$ line out the curve of intersection of the upper sheet of the hyperboloid with the plane $x^0 + x^4 = R\exp(t/R)$ (one of these is represented by the dashed line).

constant "times" $t$, the respective points in $\mathbb{H}^4$ form the intersection of the plane $x^0 + x^4 = R\exp(t/R)$ with the upper sheet of the hyperboloid.

Next, I need the line element. This can be obtained by an elementary, but clumsy calculation. I shall rather proceed by means of a transformation of the metrical tensor. Defining $y^0 := t$, the transformation equations are

$$g_{\mu\nu}(y) = \eta_{AB}\,\frac{\partial x^A}{\partial y^\mu}\,\frac{\partial x^B}{\partial y^\nu}\;, \quad (A, B = 0, \ldots, 4; \mu, \nu = 0, \ldots, 3).$$

Here I go:—

$$\begin{aligned}
g_{00} &= \left(\frac{\partial x^0}{\partial t}\right)^2 + \eta_{ij}\,\frac{\partial x^i}{\partial t}\,\frac{\partial x^j}{\partial t} - \left(\frac{\partial x^4}{\partial t}\right)^2 \\
&= \left(\mathrm{sh}\,\frac{t}{R} - \frac{\eta_{ij}\,y^i\,y^j}{2R^2}\,e^{\frac{t}{R}}\right)^2 + \frac{\eta_{ij}\,y^i\,y^j}{R^2}\,e^{2\frac{t}{R}} - \\
&\quad - \left(\mathrm{ch}\,\frac{t}{R} + \frac{\eta_{ij}\,y^i\,y^j}{2R^2}\,e^{\frac{t}{R}}\right)^2 \\
&= -1 - \frac{\eta_{ij}\,y^i\,y^j}{R^2}\,e^{\frac{t}{R}}\left(\mathrm{sh}\,\frac{t}{R} + \mathrm{ch}\,\frac{t}{R}\right) + \frac{\eta_{ij}\,y^i\,y^j}{R^2}\,e^{2\frac{t}{R}} \\
&= -1\;.
\end{aligned}$$

$$
\begin{aligned}
g_{0i} &= \eta_{AB}\,\frac{\partial x^A}{\partial t}\,\frac{\partial x^B}{\partial y^i} \\
&= \frac{\partial x^0}{\partial t}\,\frac{\partial x^0}{\partial y^i} + \eta_{kl}\,\frac{\partial x^k}{\partial t}\,\frac{\partial x^l}{\partial y^i} - \frac{\partial x^4}{\partial t}\,\frac{\partial x^4}{\partial y^i} \\
&= -\left(\mathrm{sh}\,\frac{t}{R} - \frac{\eta_{kl}\,y^k\,y^l}{2R^2}\,e^{\frac{t}{R}}\right)\cdot\frac{1}{R}\,\eta_{il}\,y^l e^{\frac{t}{R}} + \frac{1}{R}\,\eta_{il}\,y^l e^{2\frac{t}{R}} \; - \\
&\quad - \left(\mathrm{ch}\,\frac{t}{R} + \frac{\eta_{kl}\,y^k\,y^l}{2R^2}\,e^{\frac{t}{R}}\right)\cdot\frac{1}{R}\,\eta_{il}\,y^l e^{\frac{t}{R}} \\
&= -\frac{1}{R}\,\eta_{il}\,y^l e^{\frac{t}{R}}\left(\mathrm{sh}\,\frac{t}{R} + \mathrm{ch}\,\frac{t}{R}\right) + \frac{1}{R}\,\eta_{il}\,y^l e^{2\frac{t}{R}} \\
&= 0\,. \\
g_{ij} &= \eta_{AB}\,\frac{\partial x^A}{\partial y^i}\,\frac{\partial x^B}{\partial y^j} \\
&= \frac{\partial x^0}{\partial y^i}\,\frac{\partial x^0}{\partial y^j} + \eta_{kl}\,\frac{\partial x^k}{\partial y^i}\,\frac{\partial x^l}{\partial y^j} - \frac{\partial x^4}{\partial y^i}\,\frac{\partial x^4}{\partial y^j} \\
&= \frac{1}{R^2}\,\eta_{ik}\,y^k\,\eta_{jl}\,y^l\,e^{2\frac{t}{R}} + \eta_{ij}\,e^{2\frac{t}{R}} - \frac{1}{R^2}\,\eta_{ik}\,y^k\,\eta_{jl}\,y^l\,e^{2\frac{t}{R}} \\
&= \eta_{ij}\,e^{2\frac{t}{R}}\;.
\end{aligned}
$$

**Proposition 6.** *In planar co-ordinates, the metrical tensor is diagonal,*

$$
(g_{\mu\nu}) = \mathrm{diag}\left(-1, -e^{2\frac{t}{R}}, -e^{2\frac{t}{R}}, -e^{2\frac{t}{R}}\right)\;,
$$

*and the respective line element is*

$$
ds^2 = -dt^2 - e^{2\frac{t}{R}}\,\delta_{ij}\,dy^i\,dy^j \quad . \tag{35}
$$

This is of a Robertson–Walker-like form with a three-dimensional Euclidean subspace immersed in a four-dimensional *space*, not space-time.

## 6.2 The Metric in $\mathbb{H}^4$, expressed in Planar Co-Ordinates, is Conformally Euclidean

I will proceed and show:—

**Proposition 7.** *The metric in eqn* (35) *is conformal with that of the four-dimensional Euclidean space* $\mathbb{R}^4$.

*Proof.* I rewrite eqn (35) in the form

$$
ds^2 = -e^{2\frac{t}{R}}\left(e^{-2\frac{t}{R}}\,dt^2 + \delta_{ij}\,dy^i\,dy^j\right) \quad ,
$$

and introduce the “conformal time” $T$ by

$$dT := e^{-\frac{t}{R}}\, dt \quad , \tag{36}$$

which can be integrated to give

$$T = -R\, e^{-\frac{t}{R}} \quad .$$

From this it follows that

$$e^{2\frac{t}{R}} = R^2/T^2 \ , \tag{37}$$

so that

$$ds^2 = -\frac{R^2}{T^2} \left( dT^2 + \delta_{ij}\, dy^i\, dy^j \right) \quad .$$

With the “conformal time” $T$ as defined in eqn (36) and by re-defining $y^0$ as $y^0 := T$, the line element takes the required form $ds^2 = \lambda\, ds^2_{\mathrm{Euclid}}$:—

$$ds^2 = -\frac{R^2}{T^2}\, \delta_{\mu\nu}\, dy^\mu\, dy^\nu \quad (\mu, \nu = 0, \ldots 3) \ . \qquad \square$$

## 6.3 Planar Co-Ordinates in the Space-Time $dS^4$

These co-ordinates are introduced in a manner very similar to the case of the *space* $\mathbb{H}^4$ (eqn [34]):—

$$\begin{aligned} x^0 &= R\,\mathrm{sh}\,\frac{ct}{R} - \frac{1}{2R}\, \eta_{ij}\, y^i\, y^j\, e^{\frac{ct}{R}} \quad (i = 1, 2, 3) \quad , \\ x^i &= y^i\, e^{\frac{ct}{R}} \quad , \\ x^4 &= R\,\mathrm{ch}\,\frac{ct}{R} + \frac{1}{2R}\, \eta_{ij}\, y^i\, y^j\, e^{\frac{ct}{R}} \quad . \end{aligned} \tag{38}$$

The metrical tensor of $\mathbb{M}^5$ is again $(\eta_{AB}) = \mathrm{diag}(1, -1, -1, -1, -1)$. As before, I prefer to measure $t$ in this physical space-time in units of time, this is why $c$ comes in.

The difference to $\mathbb{H}^4$ lies in sh and ch being interchanged.

The loci of all events (= points) in $dS^4$ that occur at the same co-ordinate time $t$ are identical with the intersection of the hyperboloid with the plane $x^0 + x^4 = R \exp(ct/R)$, the limiting plane for $t \to -\infty$ being $x^0 + x^4 = 0$, which means that this net of co-ordinates covers only one part of $dS^4$: $x^0 + x^4 > 0$.

I can visualise this in three dimensions, in which I have the co-ordinates

$$\begin{aligned} x^0 &= R\,\mathrm{sh}\,\frac{ct}{R} + \frac{y^2}{2R}\, e^{\frac{ct}{R}} \ , \\ x^1 &= y\, e^{\frac{ct}{R}} \ , \\ x^4 &= R\,\mathrm{ch}\,\frac{ct}{R} - \frac{y^2}{2R}\, e^{\frac{ct}{R}} \ . \end{aligned}$$

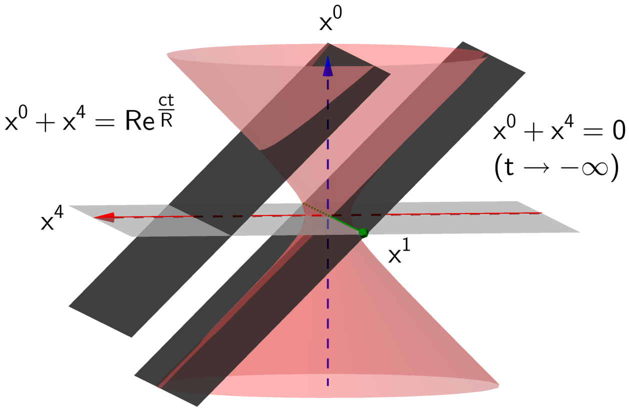


Figure 4: $dS^2$ in planar co-ordinates. Only the part of the hyperboloid is covered by these co-ordinates for which $x^0 + x^4 = R\exp(ct/R) > 0$. The loci of all simultaneous events that occur at the time $t$ form the curve of intersection of the plane $x^0 + x^4 = R\exp(ct/R)$ with the hyperboloid. Also shown is the limiting plane $x^0 + x^4 = 0$, which represents the temporal limit $t \to -\infty$.

Fig. 4 displays the respective three-dimensional aspect of $dS^2$, together with an intersecting plane

$$x^0 + x^4 = R\,e^{\frac{ct}{R}} \,, \tag{39}$$

on which a constant, positive value is assigned to $t$. Also shown is the limiting plane $t \to -\infty$. The part of $dS^2$ that, in that figure, lies to the right of this limiting plane is not covered by these co-ordinates. In fig. 5 only the $x^0x^4$ plane of $dS^2$ is shown, together with the intersecting lines of three of the planes of constant $t$ (eqn [39]), namely $t > 0$, $t = 0$, $t \to -\infty$.

That the co-ordinates of eqs (38) in fact satisfy eqn (4) will now be proven:—

**Proposition 8.** *If the five co-ordinates $x^A$ ($A = 0, \ldots, 4$) are expressed by the independent co-ordinates $t$, $y^1$, $y^2$, $y^3$ according to eqs* (38)*, they will fulfil the constraint imposed by eqn* (4)*, which is*

$$\eta_{AB}\, x^A\, x^B = -R^2 \,.$$

*Proof.*

$$\begin{aligned}\eta_{AB}\, x^A\, x^B \;&=\; \left(R\,\mathrm{sh}\,\frac{ct}{R} - \frac{1}{2R}\,\eta_{ij}\, y^i\, y^j\, e^{\frac{ct}{R}}\right)^2 + \eta_{ij}\, y^i\, y^j\, e^{2\frac{ct}{R}} \;-\\ &\quad - \left(R\,\mathrm{ch}\,\frac{ct}{R} + \frac{1}{2R}\,\eta_{ij}\, y^i\, y^j\, e^{\frac{ct}{R}}\right)^2\end{aligned}$$

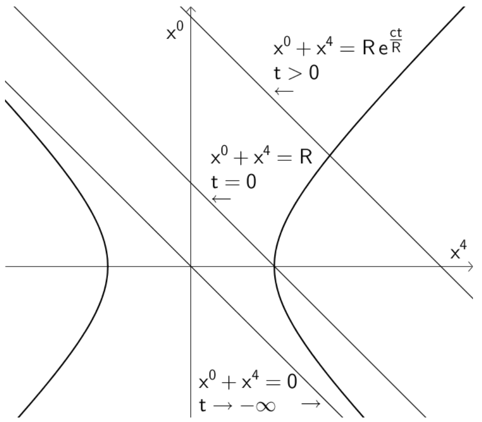


Figure 5: The $x^0 x^4$ plane of $dS^2$ in planar co-ordinates. The lines are the intersections of planes on which the co-ordinate time $t$ is of a certain constant value. These planes are subject to the equation $x^0 + x^4 = R \exp(ct/R)$. Shown are the cases $t \to -\infty$, $t = 0$, and $t > 0$.

$$\curvearrowright \quad \eta_{AB}\, x^A\, x^B = -R^2 - \eta_{ij}\, y^i\, y^j\, e^{\frac{ct}{R}} \left(\operatorname{sh}\frac{ct}{R} + \operatorname{ch}\frac{ct}{R}\right) + \eta_{ij}\, y^i\, y^j\, e^{2\frac{ct}{R}}$$
$$= -R^2 \qquad \square$$

Now I show that the metric is spatially flat in these co-ordinates:—

**Proposition 9.** *The square of the line element in $dS^4$, expressed in planar co-ordinates, is*

$$ds^2 = c^2\, dt^2 - e^{2\frac{ct}{R}}\, \delta_{ij}\, dy^i\, dy^j \,, \tag{40}$$

*which means the spatial part of the metric is conformally Euclidean.*

*Proof.* The parts of the expressions for $x^A$, given by eqs (38), that depend on the $y^i$ are the same for both $\mathbb{H}^4$ and $dS^4$, so I can refer to the derivation of eqn (35).

$$\begin{aligned} g_{00} &= \frac{1}{c^2}\, \eta_{AB}\, \frac{\partial x^A}{\partial t}\, \frac{\partial x^B}{\partial t} \\ &= \frac{1}{c^2}\left(\frac{\partial x^0}{\partial t}\right)^2 + \frac{1}{c^2}\, \eta_{ij}\, \frac{\partial x^i}{\partial t}\, \frac{\partial x^j}{\partial t} - \frac{1}{c^2}\left(\frac{\partial x^4}{\partial t}\right)^2 \\ &= \left(\operatorname{ch}\frac{ct}{R} - \frac{1}{2R^2}\, \eta_{ij}\, y^i\, y^j\, e^{\frac{ct}{R}}\right)^2 + \frac{\eta_{ij}\, y^i\, y^j}{R^2}\, e^{2\frac{ct}{R}} - \\ &\quad - \left(\operatorname{sh}\frac{ct}{R} + \frac{1}{2R^2}\, \eta_{ij}\, y^i\, y^j\, e^{\frac{ct}{R}}\right)^2 \\ &= 1\,. \end{aligned}$$

$$
\begin{aligned}
g_{0i} &= \frac{1}{c}\,\eta_{AB}\,\frac{\partial x^A}{\partial t}\,\frac{\partial x^B}{\partial y^i} \\
&= \frac{1}{c}\,\frac{\partial x^0}{\partial t}\,\frac{\partial x^0}{\partial y^i} + \frac{1}{c}\,\eta_{kl}\,\frac{\partial x^k}{\partial t}\,\frac{\partial x^l}{\partial y^i} - \frac{1}{c}\,\frac{\partial x^4}{\partial t}\,\frac{\partial x^4}{\partial y^i} \\
&= -\left(\mathrm{ch}\,\frac{ct}{R} - \frac{1}{2R^2}\,\eta_{ij}\,y^i\,y^j\,e^{\frac{ct}{R}}\right)\frac{1}{R}\,\eta_{il}\,y^l\,e^{\frac{ct}{R}} + \frac{1}{R}\,\eta_{il}\,y^l\,e^{2\frac{ct}{R}} \;- \\
&\quad - \left(\mathrm{sh}\,\frac{ct}{R} + \frac{1}{2R^2}\,\eta_{ij}\,y^i\,y^j\,e^{\frac{ct}{R}}\right)\frac{1}{R}\,\eta_{il}\,y^l\,e^{\frac{ct}{R}} \\
&= 0\,.
\end{aligned}
$$

As the spatial components of $x^0$ and $x^4$ as well as the $x^i$ themselves are the same as for $\mathbb{H}^4$, the squared line element is

$$ds^2 = c^2\,dt^2 - e^{2\frac{ct}{R}}\,\delta_{ij}\,dy^i\,dy^j\;. \qquad \square$$

## 6.4 Planar Co-Ordinates in $dS^4$ and an Expanding, Spatially Euclidean Universe

If the $y^i$ are expressed by the spherical co-ordinates $R_0\,\chi$, $\vartheta$, and $\varphi$ from the Robertson-Walker metric, the square of the line element in eqn (40) takes the form

$$ds^2 = c^2\,dt^2 - R_0^2\,e^{2\frac{ct}{R}}\left(d\chi^2 + \chi^2\,d\Omega^2\right)\;. \tag{41}$$

The scale factor $\mathsf{R}(t) = R_0 \exp\left(ct/R\right)$ coincides with the one that occurs in the exponential solution (eqn [31]) of the Friedmann equation (30) (page 11), if $k = 0$ and $R = \sqrt{3/\Lambda}$ are assumed. This implies:

**Proposition 10.** *With the introduction of the planar co-ordinates as the four independent parameters of $dS^4$, time and space are separated in such a manner as to make the universe appear to be a flat space that undergoes an eternal exponential expansion.*

The co-ordinate time $t$, here, is not the same as the one which is defined as the temporal *spherical* co-ordinate in eqs (16), although I used the same letter $t$. A description of $dS^4$ in spherical co-ordinates results in a spatially closed universe the temporal behaviour of which follows the “big-bounce” scenario. The reason for this difference lies in the fact that Einstein’s field equations and, hence, the Friedmann equations, in an empty universe, depend solely on its metric, see § 5:—

**Planar co-ordinates** imply a metric of the form of eqn (41), which is spatially Euclidean, *i. e.* $f(\chi) = \chi$ (cf. the second eqn [28] on page 10). If the Friedmann equation is derived from *these* respective metrical coefficients, $k = 0$ will be found in eqn (30), and to an observer who set his clock and adjusted his measuring rods according to the planar co-ordinates of $dS^4$, the universe will appear spatially flat and display an eternal exponential expansion.

**Spherical co-ordinates,** however, imply a metric of the form of eqn (17):—

$$ds^2 = c^2\,dt^2 - R^2\,\mathrm{ch}^2\,\frac{ct}{R}\left(d\chi^2 + \sin^2\chi\,d\Omega^2\right)\,,$$

which is spatially closed, *i. e.* $f(\chi) = \sin\chi$. If the Friedmann equation is derived from *those* metrical coefficients, $k = 1$ will be found in eqn (30). An observer whose clock is set and measuring rods are scaled according to the spherical co-ordinates of $dS^4$ will conclude that the universe is spatially closed and evolves in the "big-bounce" manner.

The evolution of the de Sitter universe, as it appears in planar co-ordinates, is sketched in three dimensions in fig. 6.

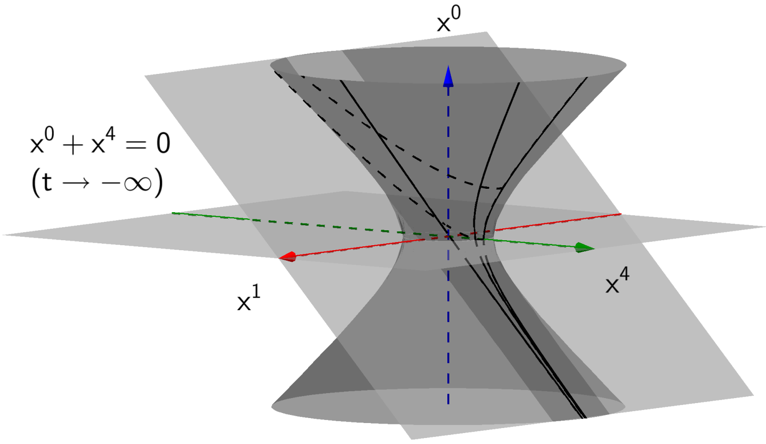


Figure 6: Evolution of the de Sitter universe, as seen in planar co-ordinates. In $dS^2$ there are only two: $t$ and $y$. The solid curves have been drawn for $y = \text{const.} \geqslant 0$ and variable $t \in ]-\infty, +\infty[$, the curve on which $y = 0$ lies in the $x^0x^4$ plane (right solid curve). The curves of intersection of $dS^2$ with the plane $x^0 + x^4 = 0$ limit the régime that is covered by these co-ordinates. The dashed curves are the loci of constant $t$ with $t = 0$ on the lower one of the two curves shown. Only $y \geqslant 0$ has been considered. As in the Robertson-Walker metric all Christoffel symbols $\Gamma^{\mu}_{00}$ are zero, the solid curves shown are geodesics: $(u^\mu) = (1, 0)^T$ for $y = \text{const.}$

## 6.5 Conformality with $\mathbb{M}^4$

**Proposition 11.** *The metric of $dS^4$ in planar co-ordinates is conformally Minkowskian, with*

$$ds^2 = \frac{R^2}{T^2}\left(c^2\,dt^2 - \delta_{ij}\,dy^i\,dy^j\right)\,,$$

*where $T$ is the conformal time defined by*

$$dT := e^{-\frac{ct}{R}}\, c\, dt\ .$$

*Proof.* I rewrite $ds^2$ from proposition 9, eqn (40), as

$$ds^2 = e^{2\frac{ct}{R}} \left( e^{-2\frac{ct}{R}}\, c^2\, dt^2 - \delta_{ij}\, dy^i\, dy^j \right)\ ,$$

and define $dT := \exp(-ct/R)\, c\, dt$ as the conformal time, just as I did in eqn (36) in the proof of proposition 7. Consequently, eqn (37),

$$e^{2\frac{ct}{R}} = \frac{R^2}{T^2}\quad ,$$

is valid in the present case, too, and the proposition is proven. □

# 7 Hyperbolic Co-Ordinates

## 7.1 $\mathbb{H}^4$

### 7.1.1 $\mathbb{H}^4$ in Hyperbolic Co-Ordinates

Similarly to what is done in the de Sitter space-time (§ 7.2), I introduce "hyperbolic co-ordinates"

$$\begin{aligned} x^\mu &= R\ \mathrm{ch}\,\frac{t}{R}\cdot\omega^\mu\ , \quad \mu = 0,\ldots,3; \\ x^4 &= R\ \mathrm{sh}\,\frac{t}{R}\ . \end{aligned} \tag{42}$$

In order to have eqn (3) hold, upon the $\omega^\mu$ the condition

$$\eta_{\mu\nu}\,\omega^\mu\omega^\nu = 1 \tag{43}$$

is imposed. I then have

$$\begin{aligned} \eta_{AB}\, x^A x^B &= \eta_{\mu\nu}\, x^\mu x^\nu - (x^4)^2 \\ &= R^2\,\mathrm{ch}^2\,\frac{t}{R}\cdot\eta_{\mu\nu}\,\omega^\mu\omega^\nu - R^2\,\mathrm{sh}^2\,\frac{t}{R} \\ &\overset{(43)}{=} R^2 \end{aligned}$$

and

$$\eta_{\mu\nu}\,\omega^\mu\, d\omega^\nu = 0\ . \tag{44}$$

A visualisation in three dimensions is shown in fig. 7 on page 26.

The metrical tensor will now be expressed by means of the $\omega^\mu$ and $t$. To this end, I define the barred co-ordinates $\bar{x}^\mu = \omega^\mu$ ($\mu = 0,\ldots,3$) and $\bar{x}^4 = t = \omega^4$.

Instead of an elementary calculation similar to that in Ch. 3, I perform a transformation of the metrical tensor. For this purpose, I regard the $\omega^\mu$ as independent, *i. e.* I will not observe the restriction of eqn (43) until at a later stage:—

$$\bar{g}_{CD} = \eta_{AB}\,\frac{\partial x^A}{\partial \omega^C}\,\frac{\partial x^B}{\partial \omega^D} \quad (A, B, C, D = 0, \dots, 4) \ .$$

$$\begin{aligned}
\bar{g}_{\mu\nu} &= \eta_{AB}\,\frac{\partial x^A}{\partial \omega^\mu}\,\frac{\partial x^B}{\partial \omega^\nu} = \eta_{\mu\nu}\,R^2\,\mathrm{ch}^2\,\frac{t}{R} \ . \\
\bar{g}_{44} &= \eta_{AB}\,\frac{\partial x^A}{\partial t}\,\frac{\partial x^B}{\partial t} \\
&= \mathrm{sh}^2\,\frac{t}{R}\,\eta_{\mu\nu}\,\omega^\mu\,\omega^\nu - \mathrm{ch}^2\,\frac{t}{R} \ . \\
\bar{g}_{4\mu} &= \eta_{AB}\,\frac{\partial x^A}{\partial t}\,\frac{\partial x^B}{\partial \omega^\mu} \\
&= \eta_{\varkappa\lambda}\,\frac{\partial x^\varkappa}{\partial t}\,\frac{\partial x^\lambda}{\partial \omega^\mu} - \frac{\partial x^4}{\partial t}\,\underbrace{\frac{\partial x^4}{\partial \omega^\mu}}_{0} \\
&= R\ \mathrm{sh}\,\frac{t}{R}\ \mathrm{ch}\,\frac{t}{R}\,\eta_{\mu\varkappa}\,\omega^\varkappa \ . \qquad (45)
\end{aligned}$$

Note that the metrical tensor is not diagonal!

The line element, expressed in these co-ordinates, is

$$ds^2 = \bar{g}_{\mu\nu}\,d\omega^\mu\,d\omega^\nu + 2\bar{g}_{4\mu}\,dt\,d\omega^\mu + \bar{g}_{44}\,dt^2 \ . \qquad (46)$$

Now I constrain the co-ordinates to the surface $\mathbb{H}^4$ by means of eqn (43) and, consequently, eqn (44), which renders the square of the line element

$$ds^2 = R^2\,\mathrm{ch}^2\,\frac{t}{R}\,\eta_{\mu\nu}\ d\omega^\mu\,d\omega^\nu\Big|_{\mathbb{H}^4} - dt^2 \quad .$$

Because of this constraint, the *four* co-ordinates $\omega^\mu$ depend on *three* independent co-ordinates, which I will designate by $\bar{\bar{x}}^i$, so that the line element may be rewritten as

$$ds^2 = -dt^2 + R^2\,\mathrm{ch}^2\,\frac{t}{R}\,\eta_{\mu\nu}\,\frac{\partial \omega^\mu}{\partial \bar{\bar{x}}^i}\,\frac{\partial \omega^\nu}{\bar{\bar{x}}^j}\,d\bar{\bar{x}}^i\,d\bar{\bar{x}}^j \quad , \qquad (47)$$

so defining the spatial metrical coefficients

$$a_{ij} = \eta_{\mu\nu}\,\frac{\partial \omega^\mu}{\partial \bar{\bar{x}}^i}\,\frac{\partial \omega^\nu}{\bar{\bar{x}}^j} \quad . \qquad (48)$$

At this point, I specify the three independent spatial co-ordinates $(\bar{\bar{x}}^i) := (\chi, \vartheta, \varphi)^T$:—

$$\begin{aligned}
\omega^0 &= \pm\,\mathrm{ch}\,\chi \ , \\
\omega^1 &= \mathrm{sh}\,\chi\,\cos\vartheta \ , \\
\omega^2 &= \mathrm{sh}\,\chi\,\sin\vartheta\,\cos\varphi \ , \\
\omega^3 &= \mathrm{sh}\,\chi\,\sin\vartheta\,\sin\varphi \ . \qquad (49)
\end{aligned}$$

If $\omega^0$ is inserted into the first eqn (42), it becomes obvious that all values of $x^0 \in\ ]-R, R[$ are exempted (hyperboloid of two sheets).

The metrical tensor in these co-ordinates is

$$(\bar{\bar{g}}_{\mu\nu}) = \left(\begin{array}{c|c} -1 & \mathbf{0}^T \\ \hline \mathbf{0} & R^2 \operatorname{ch}^2 \dfrac{t}{R}\,(a_{ij}) \end{array}\right) \quad . \tag{50}$$

To obtain the metrical coefficients, the $a_{ij}$ are required (eqn [48]):—

$$\begin{aligned}
a_{11} &= \left(\frac{\partial\omega^0}{\partial\chi}\right)^2 - \sum_{i=1}^{3}\left(\frac{\partial\omega^i}{\partial\chi}\right)^2 \\
&= \operatorname{sh}^2\chi - \operatorname{ch}^2\chi = -1\ . \\
a_{12} &= \eta_{ij}\,\frac{\partial\omega^i}{\partial\chi}\,\frac{\partial\omega^j}{\partial\vartheta} = \\
&= -\operatorname{sh}\chi\operatorname{ch}\chi\left(-\sin\vartheta\cos\vartheta + \sin\vartheta\cos\vartheta\,[\cos^2\varphi + \sin^2\varphi]\right) = 0\ . \\
a_{13} &= -\sum_{i=2}^{3}\frac{\partial\omega^i}{\partial\chi}\,\frac{\partial\omega^i}{\partial\varphi} \\
&= -\operatorname{sh}\chi\operatorname{ch}\chi\sin^2\vartheta\,(-\sin\varphi\cos\varphi + \sin\varphi\cos\varphi) = 0\ . \\
a_{22} &= -\sum_{i=1}^{3}\left(\frac{\partial\omega^i}{\partial\vartheta}\right)^2 = -\operatorname{sh}^2\chi\ . \\
a_{23} &= -\sum_{i=2}^{3}\frac{\partial\omega^i}{\partial\vartheta}\,\frac{\partial\omega^i}{\partial\varphi} \\
&= -\operatorname{sh}^2\chi\sin\vartheta\cos\vartheta\,(-\sin\varphi\cos\varphi + \sin\varphi\cos\varphi) = 0\ . \\
a_{33} &= -\sum_{i=2}^{3}\left(\frac{\partial\omega^i}{\partial\varphi}\right)^2 \\
&= -\operatorname{sh}^2\chi\sin^2\vartheta\left(\sin^2\varphi + \cos^2\varphi\right) = -\operatorname{sh}^2\chi\sin^2\vartheta\ .
\end{aligned}$$

The metrical tensor of the subspace is, therefore,

$$(a_{ij}) = -\operatorname{diag}\left(1\,,\ \operatorname{sh}^2\chi\,,\ \operatorname{sh}^2\chi\sin^2\vartheta\right)\ , \tag{51}$$

and, with this formula used in eqn (50), the line element follows:—

**Proposition 12.** *In hyperbolic co-ordinates the square of the line element in* $\mathbb{H}^4$ *is given by*

$$ds^2 = -dt^2 - R^2\operatorname{ch}^2\frac{t}{R}\left(d\chi^2 + \operatorname{sh}^2\chi\,d\Omega^2\right)\ .$$

$d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\,d\varphi^2$ *is the square of the line element in* $\mathbb{S}^2$.

From the foregoing calculations, I can, in addition, derive

**Proposition 13** (Metric inside $\mathbb{H}^3$)**.** *The metrical coefficients given in eqn* (51) *define a possible metric on the surface* $\mathbb{H}^3$ *of a hyperboloid of two sheets and unit radius, immersed in a four-dimensional Minkowski space* $\mathbb{M}^4$.

*Proof.* By their definition in eqs (49) it is clear that the $\omega^\alpha = \omega^\alpha(\chi, \vartheta, \varphi)$ are the co-ordinates of all points on the surface $\eta_{\alpha\beta}\,\omega^\alpha\,\omega^\beta = 1$, which is that of a hyperboloid of two sheets and unit radius. This proves the first part of the proposition.

That the hyperboloid is immersed in $\mathbb{M}^4$, not $\mathbb{R}^4$, follows from the fact that the resulting metrical tensor $a_{ij}$ in eqn (51) is derived from the equation

$$a_{ij} = \eta_{\alpha\beta}\,\frac{\partial\omega^\alpha}{\partial\bar{\bar{x}}^i}\,\frac{\partial\omega^\beta}{\bar{\bar{x}}^j} \quad ,$$

with $(\bar{\bar{x}}^i) = (\chi, \vartheta, \varphi)^T$, which is based on the Minkowski metric $\eta_{\alpha\beta}$, not on the Euclidean. □

**Remark.**—The line element in $\mathbb{H}^3$ that follows from the metrical tensor given in eqn (51),

$$ds_3^2 = -d\chi^2 - \mathrm{sh}^2\,\chi\left(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2\right) ,$$

is analogous to that in $\mathbb{H}^4$, expressed in spherical co-ordinates (eqn [15]), if $R = 1$ is considered:

$$ds^2 = -dt^2 - \mathrm{sh}^2\,t\left\{d\chi^2 + \sin^2\chi\left(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2\right)\right\} .$$

In the three-dimensional case, $\mathbb{H}^3$, the squared line element contains

$$d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\,d\varphi^2 ,$$

which is the square of the line element on the surface $\mathbb{S}^2$ of a sphere immersed in Euclidean $\mathbb{R}^3$, whilst in the four-dimensional case the squared line element in $\mathbb{H}^4$ contains

$$d\chi^2 + \sin^2\chi\left(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2\right) :$$

the square of the line element on $\mathbb{S}^3$ (hypersphere immersed in $\mathbb{R}^4$). The reason for this similarity is that the $\omega^\alpha$ are spherical co-ordinates, as direct comparison of eqs (49) with eqs (5) and (12) reveals.

For the sake of completeness, I will note down the line elements for two and one dimensions:—

**Two dimensions.** The spherical co-ordinates of $\mathbb{H}^2$ are

$$\omega^0 = \operatorname{ch}\chi\ , \quad \omega^1 = \operatorname{sh}\chi\cos\vartheta\ , \quad \omega^2 = \operatorname{sh}\chi\sin\vartheta\ ,$$

resulting in the line element

$$ds_2^2 = \eta_{\alpha\beta}\,\frac{\partial\omega^\alpha}{\partial x^\mu}\,\frac{\partial\omega^\beta}{\partial x^\nu}\,dx^\mu\,dx^\nu = -d\chi^2 - \operatorname{sh}^2\chi\,d\vartheta^2\ ,$$

$\alpha,\beta = 0,1,2$; $(x^\mu) = (\chi,\vartheta)^T$. Note the difference from § 7.1.2, where hyperbolic co-ordinates are used.

**One dimension.** The spherical co-ordinates of $\mathbb{H}$ are just

$$\omega^0 = \operatorname{ch}\chi\ , \quad \omega^1 = \operatorname{sh}\chi\ ,$$

which render the line element

$$ds_1^2 = \left(\frac{d\omega^0}{d\chi}\right)^2 d\chi^2 - \left(\frac{d\omega^1}{d\chi}\right)^2 d\chi^2 = -d\chi^2\ .$$

### 7.1.2 Visualisation of Hyperbolic Co-Ordinates in 3 Dimensions

In three dimensions, of the five I retain the three co-ordinates

$$\begin{aligned} x^0 &= \pm R\operatorname{ch}\frac{t}{R}\,\operatorname{ch}\chi\ , \\ x^1 &= R\operatorname{ch}\frac{t}{R}\,\operatorname{sh}\chi\ , \\ x^4 &= R\operatorname{sh}\frac{t}{R}\ . \end{aligned}$$

These obviously obey the condition of a hyperboloid of two sheets in $\mathbb{M}^3$:

$$\mathbb{H}^2 = \left\{x \in \mathbb{M}^3 \,\middle|\, (x^0)^2 - (x^1)^2 - (x^4)^2 = R^2\right\}\ .$$

The metric on this surface is just

$$ds^2 = -dt^2 - R^2\,\operatorname{ch}^2\frac{t}{R}\,d\chi^2$$

or

$$(g_{\mu\nu}) = \operatorname{diag}\left(-1\ ,\ -R^2\,\operatorname{ch}^2\frac{t}{R}\right)\ .$$

The upper sheet $x^0 \geqslant 0$ is shown in fig. 7.

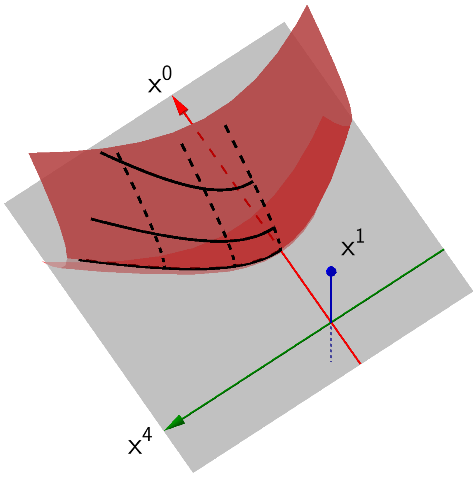


Figure 7: The sheet of $\mathbb{H}^2$ for which $x^0 \geqslant 0$. Also shown are curves of constant "time" $t \geqslant 0$ and varying $\chi \geqslant 0$ (dashed). The loci of the points with $t = 0$ are the dashed curve for which $x^4 = 0$. The solid curves are drawn for constant $\chi \geqslant 0$ and increasing "time" $t \geqslant 0$. The dashed and solid curves intersect at right angles (cf. the metrical tensor given in the text). The drawing, however, seems to contradict this, because a Minkowski space-time is sketched ($x^0$ the temporal co-ordinate), and a right angle in the Minkowskian metric is not necessarily a right angle in the Euclidean. All solid curves shown are geodesics.

### 7.1.3 Conformal Metric

I rewrite the line element of proposition 12 in the form

$$ds^2 = -\operatorname{ch}^2\frac{t}{R}\left\{\operatorname{sech}^2\frac{t}{R}\,dt^2 + R^2\left(d\chi^2 + \operatorname{sh}^2\chi\,d\Omega^2\right)\right\} \quad , \tag{52}$$

where $d\Omega^2 := d\vartheta^2 + \sin^2\vartheta\,d\varphi^2$, and define a "conformal time" $T$ by

$$dT := \operatorname{sech}\frac{t}{R}\,dt\,,$$

as I did in eqn (26) of the proof of proposition 4. Then the same calculations as those carried out in the said proof lead to the result

$$\operatorname{ch}^2\frac{t}{R} = \sec^2\frac{T}{R} \quad ,$$

and the line element is

$$ds^2 = -\sec^2\frac{T}{R}\left\{dT^2 + R^2\left(d\chi^2 + \operatorname{sh}^2\chi\,d\Omega^2\right)\right\} \quad . \tag{53}$$

The prefactor $\sec^2(T/R)$ is unbounded, as $T/R \in\, ]-\pi/2, +\pi/2[$ and $\cos(\pm\pi/2) = 0$.

**Proposition 14.** *The metric that is defined by*

$$d\mathfrak{s}^2 = -dT^2 - R^2\left(d\chi^2 + \mathrm{sh}^2\chi\, d\Omega^2\right) ,$$

*with which the metric of eqn* (53) *is conformal, is a possible metric of the four-dimensional lateral surface of a hyperbolic cylinder of infinite length immersed in* $\mathbb{M}^5$. *Its cylinder axis is spatial.*

*Proof.* The metrical tensor that underlies $d\mathfrak{s}$ is diagonal: the four-dimensional space the line element of which is $d\mathfrak{s}$ contains a three-dimensional subspace the metric of which is reflected in the squared line element

$$d\ell^2 = -R^2\, d\chi^2 - R^2\, \mathrm{sh}^2\chi\, d\Omega^2$$

and, secondly, the one-dimensional space $\mathbb{R}$, the line element of which is $dT$. These subspaces are orthogonal to each other. According to proposition 13, $d\ell$ is a possible line element in $\mathbb{H}^3$, the radius of the hyperboloid being $R$. The four-dimensional space under consideration, therefore, can be split up into a hyperboloid of two sheets $\mathbb{H}^3$ (hyperboloid in Minkowski space!) and an orthogonal straight line. That is to say it is a hyperbolic cylinder.

As to the immersion, I introduce the co-ordinates

$$\xi^\alpha := R\,\omega^\alpha \quad (\alpha = 0, \ldots, 3)\ , \quad \xi^4 = T\ ,$$

where the $\omega^\alpha$ are those listed in eqs (49). As these parameters are subject to the constraint $\eta_{\alpha\beta}\,\omega^\alpha\omega^\beta = 1$ ($\alpha = 0, 1, 2, 3$), the first four co-ordinates of all points $\left(\xi^0\,|\,\xi^1\,|\,\xi^2\,|\,\xi^3\,|\,\xi^4\right)$ are confined to the surface of the hyperboloid $\eta_{\alpha\beta}\,\xi^\alpha\xi^\beta = R^2$ of two sheets. The fifth co-ordinate, $\xi^4 = T$, however, is not constrained. So the points $(\xi^\alpha, \xi^4)$ lie on the lateral surface of a hyperbolic cylinder.

I still have to show that the metric induced by this choice of co-ordinates is the correct one. Here I make use of the orthogonality: the fact that $\xi^4 = T$ is independent of $\chi$, $\vartheta$, and $\varphi$:—

The metrical coefficients as functions of the independent co-ordinates $\chi$, $\vartheta$, $\varphi$, and $T$ are, firstly,

$$g_{ij} = \eta_{AB}\,\frac{\partial\xi^A}{\partial\bar{\bar{x}}^i}\,\frac{\partial\xi^B}{\partial\bar{\bar{x}}^j} \quad (A, B = 0, \ldots, 4)\ ,$$

where $(\bar{\bar{x}}^i) = (\chi, \vartheta, \varphi)^T$, as before. However, $\xi^4$ does not depend on any of these co-ordinates, so

$$g_{ij} = \eta_{\alpha\beta}\,\frac{\partial\xi^\alpha}{\partial\bar{\bar{x}}^i}\,\frac{\partial\xi^\beta}{\partial\bar{\bar{x}}^j} \quad (\alpha, \beta = 0, \ldots, 3)\ .$$

Because of the definition of the $\xi^\alpha$ this can be rewritten as

$$g_{ij} = R^2\,\eta_{\alpha\beta}\,\frac{\partial\omega^\alpha}{\partial\bar{\bar{x}}^i}\,\frac{\partial\omega^\beta}{\partial\bar{\bar{x}}^j} \overset{(48)}{=} R^2\, a_{ij}\ .$$

Secondly, the $\omega^\alpha$ are independent of $T$, and $\xi^4 = T$ is independent of the $\bar{\bar{x}}^i$, which means that all coefficients $g_{4i}$ vanish. Lastly, $g_{44} = 1$, so that the resulting line element is, according to eqn (51):

$$d\mathfrak{s}^2 = -dT^2 - R^2 \left( d\chi^2 + \mathrm{sh}^2 \chi \, d\Omega^2 \right) \ .$$

As the $a_{ij}$ are derived from $\eta_{AB}$, the hyperbolic cylinder is immersed in $\mathbb{M}^5$, the temporal axis being $\xi^0$, meaning that the cylinder axis $\xi^4$ is spatial. □

## 7.2 $dS^4$

### 7.2.1 $dS^4$ in Hyperbolic Co-Ordinates

Compared with $\mathbb{H}^4$, in the hyperbolic co-ordinates of $dS^4$ sh$(ct/R)$ and ch$(ct/R)$ are interchanged, nothing more. Dealing with a physical space-time, I will again regard the temporal variable $t$ as being measured in units of time. Consequently, the $x^A$ are defined as ($\alpha = 0, 1, 2, 3.$):

$$x^\alpha = R \ \mathrm{sh} \frac{ct}{R} \ \omega^\alpha \ , \quad x^4 = R \ \mathrm{ch} \frac{ct}{R} \ . \tag{54}$$

The $\omega^\alpha$ are subjected to the same constraint as in the case of $\mathbb{H}^4$:

$$\eta_{\alpha\beta} \, \omega^\alpha \, \omega^\beta = 1 \ , \tag{43}$$

so that they are defined to be the same functions of $\chi$, $\vartheta$, and $\varphi$ as formerly:

$$\begin{aligned} \omega^0 &= \mathrm{ch} \chi \ , \\ \omega^1 &= \mathrm{sh} \chi \cos\vartheta \ , \\ \omega^2 &= \mathrm{sh} \chi \sin\vartheta \cos\varphi \ , \\ \omega^3 &= \mathrm{sh} \chi \sin\vartheta \sin\varphi \ . \end{aligned} \tag{49}$$

It is obvious that the $x^A$ expressed in this way satisfy eqn (4), which means that $t, \chi, \vartheta$, and $\varphi$ so chosen are possible co-ordinates in $dS^4$.

The line element in $dS^4$ can be derived along exactly the same lines as in the case of $\mathbb{H}^4$, if in eqs (45) and (46) all sh$(ct/R)$ are consistently replaced by ch$(ct/R)$ and *vice versa*. The interchange of sh and ch has the effect that the line element is now

$$ds^2 = c^2 \, dt^2 + R^2 \, \mathrm{sh}^2 \frac{ct}{R} \, a_{ij} \, d\bar{\bar{x}}^i \, d\bar{\bar{x}}^j \ , \tag{55}$$

$(\bar{\bar{x}}^i) = (\chi, \vartheta, \varphi)^T$, instead of the expression given in eqn (47). The metrical coefficients $a_{ij}$ are defined in exactly the same way as they were in connection with $\mathbb{H}^4$,

$$a_{ij} = \eta_{\mu\nu} \frac{\partial\omega^\mu}{\partial\bar{\bar{x}}^i} \frac{\partial\omega^\nu}{\bar{\bar{x}}^j} \quad , \tag{48}$$

and as the $\omega^\alpha$ are the same functions of $\chi$, $\vartheta$, and $\varphi$ as in the case of $\mathbb{H}^4$, the tensor $a_{ij}$ is unaltered:

$$(a_{ij}) = -\operatorname{diag}\left(1\,,\ \operatorname{sh}^2\chi\,,\ \operatorname{sh}^2\chi\sin^2\vartheta\right)\ . \tag{51}$$

In consequence of this, I arrive at:

**Proposition 15** (Line element in hyperbolic co-ordinates)**.** *In hyperbolic co-ordinates, the square of the line element in $dS^4$ is given by*

$$ds^2 = c^2\,dt^2 - R^2\operatorname{sh}^2\frac{ct}{R}\left(d\chi^2 + \operatorname{sh}^2\chi\,d\Omega^2\right)\ ,$$

*$d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\,d\varphi^2$ is the square of the line element of $\mathbb{S}^2$, as before.*

The spatial part of this metric relates to the subspace $\mathbb{H}^3$, see proposition 13. This means that $dS^4$ is split up into the two subspaces *time* and an expanding $\mathbb{H}^3$ by the introduction of hyperbolic co-ordinates. *Spherical* co-ordinates split up $dS^4$ into a—different—one-dimensional subspace which denotes that respective co-ordinate time—and an expanding $\mathbb{S}^3$, see proposition 2, whilst *planar* co-ordinates imply that still another one-dimensional subspace, a third and, again, different co-ordinate time, is separated from an expanding $\mathbb{R}^3$ (proposition 9).

### 7.3 Comparison with the Friedmann Equations

If the Friedmann equation (30) is derived from this metric, $k = -1$ will be found: an open universe, the scale factor of which is

$$\mathsf{R}(t) = R\operatorname{sh}\frac{ct}{R}\ ,\quad R = \sqrt{\frac{3}{\Lambda}}\ .$$

The hyperbolic co-ordinates pave another way of splitting up the independent co-ordinates of $dS^4$ into time and space: (i) spherical (or global), (ii) planar, (iii) hyperbolic co-ordinates. Although the respective co-ordinate times are always designated by $t$, they all have different meanings from each other, as they belong to totally different co-ordinate systems (see end of last paragraph). This reflects in the different forms the Friedmann equation (30) attains, according as which particular co-ordinate time is used as the—one and only—independent variable in the equation: the parameter $k$ will take the values +1, 0, or −1.

- An observer who set his clock according to the co-ordinate time and adjusted his yardsticks in accord with the spatial axes of the hyperbolic co-ordinate system will find, if he studies the universe, that there was a big bang, the universe is expanding in proportion with $\operatorname{sh}(ct/R)$, and that the spatial subspace is of constant negative curvature.

The aspect of the universe depends upon the co-ordinates chosen. There is no preferred system of co-ordinates in the empty de Sitter space. The big bang is only a co-ordinate singularity specific to hyperbolic co-ordinates.

### 7.3.1 Visualisation in Three Dimensions

To describe $dS^2$, I retain the co-ordinates

$$x^0 = R \,\mathrm{sh}\,\frac{ct}{R}\,\mathrm{ch}\chi\,,$$

$$x^1 = R \,\mathrm{sh}\,\frac{ct}{R}\,\mathrm{sh}\chi\,,$$

$$x^4 = R \,\mathrm{ch}\,\frac{ct}{R}\,.$$

These co-ordinates obviously satisfy the condition $(x^0)^2-(x^1)^2-(x^4)^2 = -R^2$. The metrical tensor is

$$\left(g_{ij}\right) = \mathrm{diag}\left(1\,,\ -R^2\,\mathrm{sh}^2\,\frac{ct}{R}\right)\,.$$

It is obvious that, for $t > 0$, $x^0 - x^1 \searrow 0$, if $\chi \to \infty$. This means that the plane $\mathscr{E}\,:\ x^0 - x^1 = 0$ limits the régime which is covered by the hyperbolic co-ordinates: the family of parameter curves $\chi$ = const. $\geqslant 0$ will have as their liming curve for $\chi \to \infty$ the curve of intersection of $\mathscr{E}$ with $dS^2$, this is shown in fig. 8.

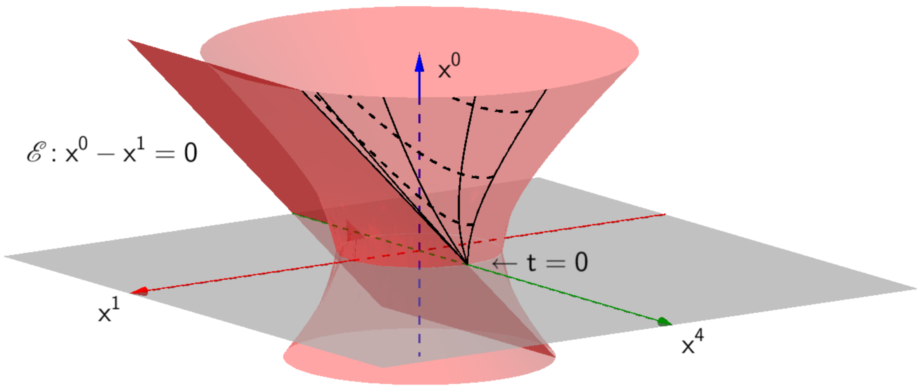


Figure 8: de Sitter space $dS^2$ immersed in $\mathbb{M}^3$. Shown are curves in hyperbolic co-ordinates with constant $\chi \geqslant 0$ and varying $t \geqslant 0$ (solid). These curves originate at $(0\,|\,0\,|\,R)$ for $t = 0$: the "big bang". Dashed curves are the loci of constant time with $\chi \geqslant 0$ regarded as variable. The régime of coverage of $dS^2$ by hyperbolic co-ordinates is limited by the curve of intersection with the plane $\mathscr{E}\,:\ x^0 - x^1 = 0$.

The solid curves in fig. 8 are drawn for varying time $t \geqslant 0$ and constant $\chi \geqslant 0$, beginning with $\chi = 0$ on the right ($x^1 \equiv 0$) and terminating with the limiting curve of intersection of $\mathscr{E}$ with $dS^2$. All these curves have their origin at $t = 0$ at the point $(0\,|\,0\,|\,R)$: the "big bang". Dashed curves are loci of constant time and variant $\chi \geqslant 0$. In the Robertson-Walker metric, the parameter curves $x^i$ = const., only $t$ varies, are geodesics. The respective 4-velocity or tangent vector is $(u^\mu) = (1, 0, 0, 0)^T$, and it satisfies the equation of a geodesic

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\varrho\sigma} u^\varrho u^\sigma = 0 \, ,$$

because all $\Gamma^\mu_{00}$ vanish in the Robertson-Walker metric, see § 8.

# 8 Geodesics of the Robertson-Walker Metric

Spherical, static, and hyperbolic co-ordinates of $dS^4$ imply that the metric within this space-time is of the Robertson-Walker kind,

$$ds^2 = c^2\, dt^2 - \mathsf{R}^2(t) \left( d\chi^2 + f^2(\chi)\, d\Omega^2 \right) \, ,$$

albeit with different functional forms of $\mathsf{R} = \mathsf{R}(t)$ and $f(\chi)$. The following remarks, therefore, apply to the three said co-ordinate systems.

I will only consider such geodesic curves as lie within the $t\chi$ plane, which shortens the line-element to

$$ds^2 = c^2\, dt^2 - \mathsf{R}^2(t)\, d\chi^2 \, .$$

The tangent vectors (or 4-velocities of free test particles) will then be of the form $(u^\mu) = (ct', \chi', 0, 0)^T$, where the accent denotes the derivative with respect to $s$. The equations of a geodesic of this kind are:—

$$\begin{array}{lcll} ct'' & + & \Gamma^0_{11}\chi'^2 & = \; 0 \, , \\ \chi'' & + & 2\Gamma^1_{01}\, c\, t'\, \chi' & = \; 0 \, , \end{array} \tag{56}$$

where

$$\Gamma^0_{11} = \frac{1}{c} \mathsf{R}\dot{\mathsf{R}} \quad \text{and} \quad \Gamma^1_{01} = \frac{1}{c} \frac{d \ln \mathsf{R}}{dt}$$

are the only non-vanishing Christoffel symbols in these equations.

If $\chi$ = const. so that $ds = c\, dt$, eqs (56) are obviously trivially fulfilled. So it follows:

**Proposition 16.** *The parameter curves of $t$ with $(u^\mu) \equiv (1, 0, 0, 0)^T$ (only $t$ varies, the other co-ordinates are held constant) are geodesics of the Robertson-Walker metric.*

This means that all the solid curves in figs 6 and 8, which actually have been drawn for varying $t$ and constant $\chi$, are geodesics. However:

**Proposition 17.** *Parameter curves of $\chi$, on which $t$ is a constant all along, whilst $\chi$ is varying, are no geodesics of the Robertson-Walker metric.*

*Proof.* Such a curve has the line element $ds = i\,\mathsf{R}\,d\chi$ and $(u^\mu) = (0, -i/\mathsf{R}, 0, 0)^T$ everywhere as its tangent vector. This would imply $\chi' \equiv$ const., $t' \equiv 0$ and hence $t'' = 0$, which is not a solution of the first eqn (56). □

The dashed curves in figs 6 and 8, which refer to constant $t$ and varying $\chi$, are no geodesics.

**Null Geodesics: Visualisation in $dS^2$**

In two dimensions, the line element is just

$$ds^2 = c^2\,dt^2 - \mathsf{R}^2(t)\,d\chi^2 \ .$$

With $ds = 0$ along a light-path, which is a null geodesic, the co-ordinate time $t$ can be utilised as a curve parameter:[¶]

$$\frac{d\chi}{dt} = \frac{c}{\mathsf{R}(t)} \quad .$$

This can easily be integrated. For hyperbolic co-ordinates, $\mathsf{R}(t) = R\,\mathrm{sh}\,\frac{ct}{R}$, hence:

$$\chi(t) = \frac{c}{R}\int_{t_1}^{t} \mathrm{csch}\,\frac{ct}{R}\,dt = \ln\frac{\mathrm{th}\,\frac{ct}{2R}}{\mathrm{th}\,\frac{ct_1}{2R}} \quad .$$

The result is shown in fig 9.

# 9 Static Co-Ordinates

## 9.1 $\mathbb{H}^4$

In analogy with the de Sitter space-time (§ 9.2), I introduce "static" co-ordinates by

$$\begin{aligned} x^0 &= \pm\sqrt{R^2 + r^2}\ \mathrm{ch}\,\frac{t}{R}\ , \\ x^4 &= \sqrt{R^2 + r^2}\ \mathrm{sh}\,\frac{t}{R}\ , \\ x^i &= r\,\omega^i \quad (i = 1, 2, 3)\ . \end{aligned} \tag{57}$$

[¶]As $t$ is not an affine parameter, the equations of a geodesic attain the form

$$\ddot{x}^\mu + \Gamma^\mu_{\varrho\sigma}\,\dot{x}^\varrho\,\dot{x}^\sigma = \frac{1}{c}\,\Gamma^0_{\varrho\sigma}\,\dot{x}^\varrho\,\dot{x}^\sigma\,\dot{x}^\mu \quad ,$$

where $t = x^0/c$ is the co-ordinate time.

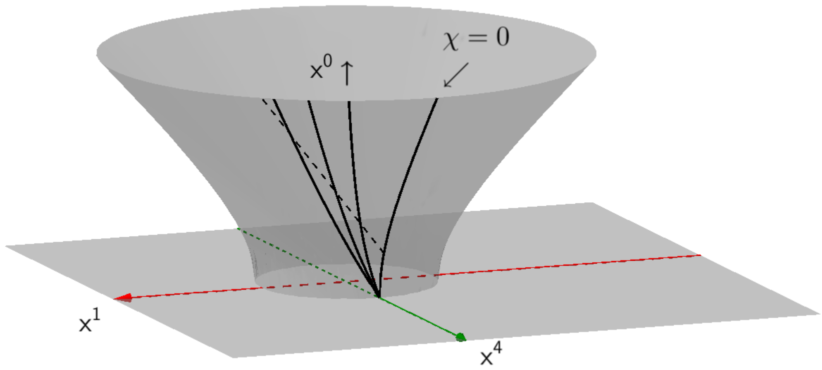


Figure 9: de Sitter space $dS^2$ in hyperbolic co-ordinates. Four world-lines of observers who are at rest ($\chi$ = const.) in the expanding frame are shown (solid). The dashed line is the null geodesic followed by a light-signal that was sent out at a certain time from the observer at $\chi = 0$ in the direction of $\chi > 0$.

In the de Sitter space-time $dS^4$, the respective radicand that appears in the static co-ordinates *there* is $R^2 - r^2$, see § 9.2, so unlike in $dS^4$, there is no horizon at $r = R$ in $\mathbb{H}^4$.

If I impose upon the $\omega^i$ the constraint ($i = 1, 2, 3.$):

$$\eta_{ij}\, \omega^i\, \omega^j = -1 \ ,$$

from which $\eta_{ij}\, \omega^i\, d\omega^j = 0$ follows, the $x^A$ will obey eqn (3).

*Proof.*

$$\begin{aligned}
\eta_{AB}\, x^A\, x^B &= (x^0)^2 + r^2\, \eta_{ij}\, \omega^i \omega^j - (x^4)^2 \\
&= \left(R^2 + r^2\right) \mathrm{ch}^2\, \frac{t}{R} - r^2 - \left(R^2 + r^2\right) \mathrm{sh}^2\, \frac{t}{R} \\
&= \left(R^2 + r^2\right) \left(\mathrm{ch}^2\, \frac{t}{R} - \mathrm{sh}^2\, \frac{t}{R}\right) - r^2 \\
&= R^2 \ .
\end{aligned}$$

□

Eqs (57) together with the restriction $\eta_{ij}\, \omega^i\, \omega^j = -1$ impose two constraints upon the five co-ordinates $x^A$:

1. $x^0$ and $x^4$ are confined to the hyperbola of radius $\sqrt{R^2 + r^2}$:

$$(x^0)^2 - (x^4)^2 = R^2 + r^2 \ ,$$

2. the $\omega^i$ and $x^i$ are, respectively, confined to the surface $\mathbb{S}^2$ of a sphere in $\mathbb{R}^3$ of unit radius and to one of radius $r$:

$$\sum_{i=1}^{3} (\omega^i)^2 = 1 \ , \qquad \sum_{i=1}^{3} (x^i)^2 = r^2 \ .$$

**Proposition 18** (Line element)**.** *In these "static" co-ordinates, the line element can be expressed in the form*

$$ds^2 = -\left(1 + \frac{r^2}{R^2}\right) dt^2 - \frac{dr^2}{1 + \dfrac{r^2}{R^2}} - r^2\, d\Omega^2 \ , \tag{58}$$

*where* $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$.

*Proof.* I shall carry through an elementary calculation here. I need the differentials of the $x^A$:—

$$\begin{aligned}
(dx^0)^2 &= \left(\frac{r\,dr}{\sqrt{R^2+r^2}}\ \mathrm{ch}\,\frac{t}{R} + \sqrt{R^2+r^2}\,\frac{1}{R}\ \mathrm{sh}\,\frac{t}{R}\,dt\right)^2 \\
&= \frac{r^2\,\mathrm{ch}^2\,\dfrac{t}{R}}{R^2+r^2}\,dr^2 + \frac{2r}{R}\ \mathrm{sh}\,\frac{t}{R}\,\mathrm{ch}\,\frac{t}{R}\,dr\,dt + \left(1+\frac{r^2}{R^2}\right)\mathrm{sh}^2\,\frac{t}{R}\,dt^2 \ , \\
(dx^4)^2 &= \left(\frac{r\,dr}{\sqrt{R^2+r^2}}\ \mathrm{sh}\,\frac{t}{R} + \sqrt{R^2+r^2}\,\frac{1}{R}\ \mathrm{ch}\,\frac{t}{R}\,dt\right)^2 \\
&= \frac{r^2\,\mathrm{sh}^2\,\dfrac{t}{R}}{R^2+r^2}\,dr^2 + \frac{2r}{R}\ \mathrm{sh}\,\frac{t}{R}\,\mathrm{ch}\,\frac{t}{R}\,dr\,dt + \left(1+\frac{r^2}{R^2}\right)\mathrm{ch}^2\,\frac{t}{R}\,dt^2 \ .
\end{aligned}$$

Subtracting:

$$(dx^0)^2 - (dx^4)^2 = \frac{r^2\,dr^2}{R^2+r^2} - \left(1+\frac{r^2}{R^2}\right)dt^2 \ .$$

From this I further have to subtract, while observing the constraint imposed on the $\omega^i$,

$$\begin{aligned}
\sum_{i=1}^{3} (dx^i)^2 &= \sum_{i=1}^{3} \left(\omega^i\,dr + r\,d\omega^i\right)^2 \\
&= \underbrace{-\eta_{ij}\,\omega^i\,\omega^j}_{1}\,dr^2 - 2\,r\,dr\,\underbrace{\eta_{ij}\,\omega^i\,d\omega^j}_{0} - r^2\,\eta_{ij}\,d\omega^i\,d\omega^j\Big|_{\mathbb{S}^2} \\
&= dr^2 - r^2\,\eta_{ij}\,d\omega^i\,d\omega^j\Big|_{\mathbb{S}^2} \\
&= dr^2 + r^2\,d\Omega^2 \ .
\end{aligned}$$

The line element is, thus:

$$ds^2 = -\left(1 + \frac{r^2}{R^2}\right) dt^2 - \frac{dr^2}{1 + \dfrac{r^2}{R^2}} - r^2\, d\Omega^2 \ .$$

On introducing the usual spherical co-ordinates on $\mathbb{S}^2$,

$$\begin{aligned}
\omega^1 &= \cos\vartheta \ , \\
\omega^2 &= \sin\vartheta \cos\varphi \ , \\
\omega^3 &= \sin\vartheta \sin\varphi \ ,
\end{aligned}$$

$d\Omega^2$ attains the familiar form

$$d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2 \ . \qquad \square$$

## 9.2 $dS^4$

### 9.2.1 $dS^4$ in Static Co-Ordinates

In a manner similar to $\mathbb{H}^4$ (eqn [57]), I introduce as co-ordinates

$$\begin{aligned}
x^0 &= \sqrt{R^2 - r^2}\ \operatorname{sh}\frac{ct}{R} \ , \\
x^4 &= \sqrt{R^2 - r^2}\ \operatorname{ch}\frac{ct}{R} \ , \\
x^i &= r\,\omega^i \quad (i = 1, 2, 3) \ . \qquad (59)
\end{aligned}$$

The radicand implies $r \in [0, R]$. The singularity at $r = R$, which is absent in $\mathbb{H}^4$, is known as the *de Sitter horizon*.

As in $\mathbb{H}^4$, upon the $\omega^i$ the constraint ($i = 1, 2, 3$):

$$\eta_{ij}\, \omega^i\, \omega^j = -1 \ ,$$

is imposed, from which $\eta_{ij}\, \omega^i\, d\omega^j = 0$ follows. Owing to the different definition of the co-ordinates as compared with $\mathbb{H}^4$, these $x^A$ will obey eqn (4).

*Proof.*

$$\begin{aligned}
\eta_{AB}\, x^A\, x^B &= (x^0)^2 + r^2\, \eta_{ij}\, \omega^i \omega^j - (x^4)^2 \\
&= \left(R^2 - r^2\right) \operatorname{sh}^2\frac{ct}{R} - r^2 - \left(R^2 - r^2\right) \operatorname{ch}^2\frac{ct}{R} \\
&= \left(R^2 - r^2\right)\left(\operatorname{sh}^2\frac{ct}{R} - \operatorname{ch}^2\frac{ct}{R}\right) - r^2 \\
&= -R^2 \ . \qquad \square
\end{aligned}$$

### 9.2.2 The Part of $dS^4$ covered by Static Co-Ordinates

Both the sum and the difference of $x^4$ and $x^0$ are positive:—

$$\begin{aligned} x^0 + x^4 &= \sqrt{R^2 - r^2}\, e^{\frac{ct}{R}} \geqslant 0 \,, \\ -x^0 + x^4 &= \sqrt{R^2 - r^2}\, e^{-\frac{ct}{R}} \geqslant 0 \,. \end{aligned}$$

Consequently, the region of $dS^4$ that is covered by static co-ordinates is bordered by the curves of intersection of $dS^4$ with the two planes $x^0 + x^4 = 0$ and $-x^0 + x^4 = 0$, see fig. 10.

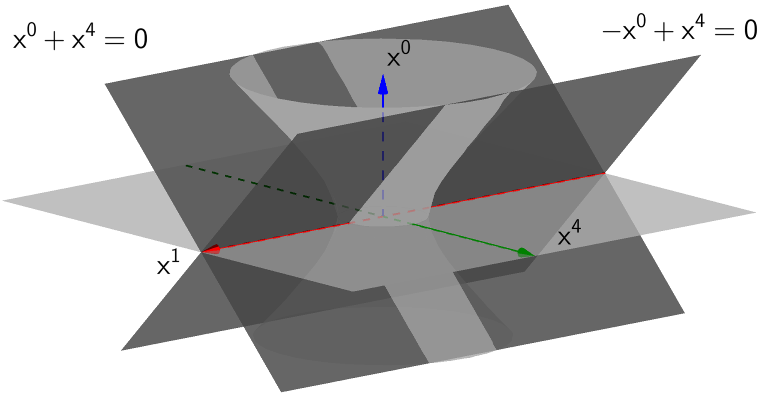


Figure 10: The region of the de Sitter space-time that is covered by static co-ordinates. This region is bounded by the curves of intersection of $dS^2$ with the two planes $x^0 + x^4 = 0$ and $-x^0 + x^4 = 0$. $x^4 \geqslant 0$ and $|x^1| \leqslant R$.

### 9.2.3 The Metric of $dS^4$ in Static Co-Ordinates

**Proposition 19** (Line element in $dS^4$ in static co-ordinates)**.** *The square of the line element within $dS^4$, expressed in static co-ordinates, is*

$$ds^2 = \left(1 - \frac{r^2}{R^2}\right) c^2\, dt^2 - \frac{dr^2}{1 - \dfrac{r^2}{R^2}} - r^2\, d\Omega^2 \,.$$

*$d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$ denotes the line element of $\mathbb{S}^2$, as usual.*

This justifies the name “static co-ordinates”, because the metrical coefficients are independent of time.

*Proof of the proposition.* I calculate the squares of the differentials $dx^0$ and $dx^4$:—

$$
\begin{aligned}
(dx^0)^2 &= \left(-\frac{r\,dr}{\sqrt{R^2-r^2}}\ \mathrm{sh}\,\frac{ct}{R} + \frac{1}{R}\ \sqrt{R^2-r^2}\ \mathrm{ch}\,\frac{ct}{R}\,c\,dt\right)^2 \\
&= \frac{r^2\,dr^2}{R^2-r^2}\ \mathrm{sh}^2\,\frac{ct}{R} - 2\,\frac{r}{R}\ \mathrm{sh}\,\frac{ct}{R}\ \mathrm{ch}\,\frac{ct}{R}\,c\,dt\,dr\ + \\
&\quad + \left(1-\frac{r^2}{R^2}\right)\mathrm{ch}^2\,\frac{ct}{R}\,c^2\,dt^2\ .
\end{aligned}
$$

Squaring $dx^4$ gives the same expression, only with sh and ch interchanged. Their difference is, thus:

$$
(dx^0)^2 - (dx^4)^2 = -\frac{r^2\,dr^2}{R^2-r^2} + \left(1-\frac{r^2}{R^2}\right)c^2\,dt^2 \quad .
$$

From this I have to subtract

$$
\begin{aligned}
\sum_{i=1}^{3}(dx^i)^2 &= \sum_{i=1}^{3}\left(\omega^i\,dr + r\,d\omega^i\right)^2 \\
&= \underbrace{-\eta_{ij}\,\omega^i\,\omega^j}_{1}\,dr^2 - 2r\,dr\ \underbrace{\eta_{ij}\,\omega^i\,d\omega^j}_{0} + r^2\ \underbrace{\delta_{ij}\,d\omega^i\,d\omega^j}_{d\Omega^2} \\
&= dr^2 + r^2\,d\Omega^2\ ,
\end{aligned}
$$

after constraining the $\omega^i$ to $\mathbb{S}^2$. The result is

$$
ds^2 = \left(1-\frac{r^2}{R^2}\right)c^2\,dt^2 - \frac{dr^2}{1-\dfrac{r^2}{R^2}} - r^2\,d\Omega^2\ .
$$

□

Static co-ordinates bring about yet another possibility of separating time from space among the four independent co-ordinates in $dS^4$. An observer at the origin who has set his clock and calibrated his yardsticks according to time and space as they appear in these co-ordinates, will find out by measurement that the de Sitter universe is static.—He will, however, also find that there are no free test-particles which are permanently at rest, see next paragraph.

### 9.2.4 Behaviour of Test Particles in the Static Metric of $dS^4$

This subject has been extensively dealt with by Tolman in *Relativity, Thermodynamics, and Cosmology*, § 144 f. I will only summarise the main facts and contemplate purely radial motion. I will not adhere to Tolman's notation.

The metric in $dS^4$, in static co-ordinates, is

$$
ds^2 = f^2\,c^2\,dt^2 - h^2\,dr^2 - r^2\,d\Omega^2\ , \qquad (60)
$$

where

$$f^2 = \frac{1}{h^2} = 1 - \frac{r^2}{R^2} \quad .$$

The equations of a geodesic, or, seen from the point of view of physics, the equations of motion of a force-free test particle,

$$x^{\mu\prime\prime} + \Gamma^{\mu}_{\varrho\sigma}\, x^{\varrho\prime}\, x^{\sigma\prime} = 0 \; ,$$

take the form

$$r'' + \frac{d\ln h}{dr}\, r'^2 - \frac{r}{h^2}\, \vartheta'^2 - \frac{r\sin^2\vartheta}{h^2}\, \varphi'^2 + \frac{f}{h^2}\, \frac{df}{dr}\, c^2 t'^2 = 0 \; ,$$

$$\vartheta'' + \frac{2}{r}\, r'\vartheta' - \sin\vartheta\cos\vartheta\, \varphi'^2 = 0 \; ,$$

$$\varphi'' + \frac{2}{r}\, r'\varphi' + 2\,\mathrm{ctg}\,\vartheta\, \vartheta'\varphi' = 0 \; ,$$

$$t'' + 2\,\frac{d\ln f}{dr}\, r't' = 0 \; .$$

The respective Christoffel symbols can be looked up in Tolman's book or derived as an exercise.

Now I will restrict myself to the plane $\vartheta = \pi/2$. The second equation of a geodesic shows that, if, additionally, $\vartheta' = 0$ is imposed, $\vartheta'' = 0$ will follow, and the respective geodesic—or path of motion of a test particle—will be confined to the said plane. The equations then simplify to

$$r'' + \frac{d\ln h}{dr}\, r'^2 - \frac{r}{h^2}\, \varphi'^2 + \frac{f}{h^2}\, \frac{df}{dr}\, c^2 t'^2 = 0 \; ,$$

$$\varphi'' + \frac{2}{r}\, r'\varphi' = 0 \; ,$$

$$t'' + 2\,\frac{d\ln f}{dr}\, r't' = 0 \; .$$

A first integral is provided by the line element:

$$f^2 c^2 t'^2 - h^2 r'^2 - r^2 \varphi'^2 - 1 = 0 \; . \tag{61}$$

A second one is obtained from the second equation:

$$\frac{d}{ds} \ln r^2 \varphi' \;=\; 0 \quad \curvearrowright$$

$$\varphi' \;=\; \frac{\ell}{r^2} \quad . \tag{62}$$

This is the conservation of the angular momentum. Lastly, from the third equation, I obtain the conservation of energy:

$$\begin{aligned} \frac{d}{ds}\ln\left(ct' f^2\right) &= 0 \quad \curvearrowright \\ ct' &= \frac{F}{f^2} \quad . \end{aligned} \tag{63}$$

There are two constants of motion: $F = f^2 ct' = g_{00}\, x^{0'} = u_0$ and $\ell$, similar to the Schwarzschild metric.

Eqs (61), (62), and (63) combine to the equations of motion of first order of a test particle:

$$\begin{aligned} r' &= \pm\sqrt{F^2 - 1 + \frac{r^2}{R^2} - \frac{\ell^2}{r^2} + \frac{\ell^2}{R^2}} \quad , \\ \varphi' &= \frac{\ell}{r^2} \quad , \\ ct' &= \frac{F}{1 - \dfrac{r^2}{R^2}} \quad . \end{aligned} \tag{64}$$

From now on I will consider only radially moving test particles, for which is $\ell = 0$, so that eqs (64) simplify to

$$\begin{aligned} r' &= \pm\sqrt{F^2 - 1 + \frac{r^2}{R^2}} \quad , \\ ct' &= \frac{F}{1 - \dfrac{r^2}{R^2}} \quad . \end{aligned} \tag{65}$$

As the co-ordinate time $t$ coincides with the proper time $s/c$ of an observer at rest at the origin ($r = 0$), I will express the velocity of a test particle in terms of $t$ rather than $s$. To this end I combine the two eqs (65) to

$$\dot{r} = \pm\frac{c}{F}\left(1 - \frac{r^2}{R^2}\right)\sqrt{F^2 - 1 + \frac{r^2}{R^2}} \quad . \tag{66}$$

An interesting result of this is that a test particle can come to a standstill at only two locations:

$$(i) \quad r^2 = R^2(1 - F^2)\ , \qquad (ii) \quad r = R\ ,$$

the first of which requires $F^2 < 1$.

However, a particle that comes to rest at $r = R\sqrt{1 - F^2}$ will not stay at rest unless $F = 1$, which implies that the particle reposes at the origin. All others that come to a momentary standstill at $r \in ]0, R[$ will be subject to an acceleration *away from the origin*.

To show this, I will compute the acceleration in terms of $t$. Differentiating eqn (66) gives:

$$\ddot{r} = \mp \frac{2r\dot{r}}{R^2} \frac{c}{F} \sqrt{F^2 - 1 + \frac{r^2}{R^2}} \pm \left(1 - \frac{r^2}{R^2}\right) \frac{c}{F} \frac{1}{R^2} \frac{\dot{r}r}{\sqrt{F^2 - 1 + \frac{r^2}{R^2}}} \quad .$$

Unfortunately, the second summand in this equation gives $\frac{0}{0}$ at $r = R\sqrt{1 - F^2}$, so I have to replace $\dot{r}$ by means of eqn (66):—

$$\ddot{r} = \mp \frac{2r\dot{r}}{R^2} \frac{c}{F} \sqrt{F^2 - 1 + \frac{r^2}{R^2}} + \left(1 - \frac{r^2}{R^2}\right)^2 \frac{c^2 r}{F^2 R^2} \quad .$$

This reveals that the spot $r = R\sqrt{F^2 - 1}$ is only a turning point of motion, no resting point: $\dot{r} = 0$ there, however, $\ddot{r} > 0$.

- An observer who rests at the origin, with his clocks set according to the co-ordinate time and yard sticks scaled according to the co-ordinate lengths induced within $dS^4$ by the static co-ordinates, will find that all particles with $F \in ]0, 1[$ are scattered away from him.

This is the "de Sitter effect".

There are only two locations at which a test particle can come to rest for good: (i) the origin (which requires $F = 1$), (ii) at $r = R$, but that is at the *de Sitter horizon*: from such a particle the observer will gain no information, see proposition 20.

### 9.2.5 de Sitter Horizon

**Proposition 20** (de Sitter horizon)**.** *An infinite interval of time, measured in the proper time $t$ of an observer who rests at the origin, is necessary for a light-signal to travel between $r = 0$ and $r = R$.*

*Proof.* I go back to the line element of eqn (60), which I rewrite for $d\Omega = 0$:

$$ds^2 = f^2\, c^2\, dt^2 - h^2\, dr^2 \ .$$

As a light path is considered, $ds = 0$, so that

$$\frac{dr}{dt} = \pm f^2 c = \pm c \left(1 - \frac{r^2}{R^2}\right) ,$$

which can easily be integrated (dropping the − sign):—

$$t = \frac{1}{c} \int_0^r \frac{dr}{1 - \frac{r^2}{R^2}} = \frac{R}{c} \operatorname{arth} \frac{r}{R} \xrightarrow[r \to R]{} \infty \ . \qquad \square$$

Similarly to the Schwarzschild horizon, $f$, the velocity of light, drops to zero at the de Sitter horizon.