Behaviour of Test Particles in the de Sitter Universe

David Walker

Contents

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Abstract

The four-velocities of two kinds of test particle are considered: such as are (i) at rest within the static and (ii) freely falling, but spatially at rest, in Robertson-Walker co-ordinates. The former are not moving on geodesic world-lines, whilst the latter do. I will express their respective four-velocities as well as the four-acceleration of the former in both the static and the Robertson-Walker co-ordinates. I visualise the respective geodesic curves in $dS^2 \subset \mathbb{M}^3$. Special emphasis is laid on the event horizon.
(Typed in $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX, graphics produced with GeoGebra)

1 Introduction of the Co-Ordinates

1.1 The Two Systems

Static. Bare symbols refer to the static co-ordinates $(x^\mu) = (ct, r, \vartheta, \varphi)$. In these, the line element is given by

$$ds^2 = f^2\, c^2\, dt^2 - h^2\, dr^2 - r^2\, d\Omega^2 \quad , \tag{1}$$

where

$$f^2 = \frac{1}{h^2} = 1 - \frac{r^2}{a^2} \quad . \tag{2}$$

$d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$ is the square of the line element on the surface of the unit sphere $\mathbb{S}^2$, as usual.

Robertson-Walker. Overlined symbols will refer to the R. W. co-ordinates $(\bar{x}^\mu) = (c\bar{t}, \bar{r}, \bar{\vartheta}, \bar{\varphi})$. Instead of $\bar{r}$, I will use the symbol χ.* The line element takes the R. W. form

$$ds^2 = c^2\, d\bar{t}^2 - R^2(\bar{t})\left(d\chi^2 + \chi^2\, d\Omega^2\right) \quad , \tag{3}$$

where

$$R(\bar{t}) = e^{\frac{c\,\bar{t}}{a}} \quad , \tag{4}$$

*Unlike in other papers, χ is not dimensionless here, instead, it has the dimension of a length.

after the introduction of the co-ordinates

$$\begin{aligned} c\,\bar{t} &= c\,t + a\,\ln f \quad , \\ \chi \equiv \bar{r} &= r\,h\,e^{-\frac{c\,t}{a}} \quad , \end{aligned} \tag{5}$$

see Tolman, p. 347.[†]

1.2 The Inverse Transformation

Proposition 1. *The transformation from the R. W. co-ordinates $\bar{x}^\mu$ back to the static co-ordinates x^μ is accomplished by*

$$\begin{aligned} c\,t &= c\,\bar{t} - \frac{1}{2}\,a\ln\left(1 - \frac{\chi^2}{a^2}\,e^{\frac{2c\bar{t}}{a}}\right) \quad , \\ r &= \chi\,e^{\frac{c\bar{t}}{a}} \quad , \end{aligned}$$

with $\chi \equiv \bar{r}$.

Proof. I re-write the first eqn (5) in the form

$$e^{c\,\frac{\bar{t}-t}{a}} = f = \frac{1}{h} \quad \Leftrightarrow$$

$$e^{-\frac{c\,\bar{t}}{a}} = h\,e^{-\frac{c\,t}{a}} \quad .$$

According to the second eqn (5), the r. h. s. of the last equation is just χ/r, so that

$$r = \chi\,e^{\frac{c\,\bar{t}}{a}}$$

follows. This proves the second equation of the proposition.

The first equation to be verified follows from the first eqn (5) by solving it for $c\,t$ and then replace r by the relation just proven. □

2 Four-Velocity and Acceleration of a Test Particle at Rest with respect to the Static Co-Ordinates

2.1 Static Co-Ordinates

The spatial co-ordinates of such a particle remain unchanged, i. e. $dr = d\Omega = 0$, and from the line element of eqn (1) I have

$$1 = f^2\left(\frac{dct}{ds}\right)^2 \equiv f^2\,(u^0)^2 \quad .$$

[†]Tolman uses the letter R instead of a. I do not follow him here, because I always denote by R the scale factor of the R. W. M. All references are to *Relativity, Thermodynamics, and Cosmology* by R. Tolman.

In consequence of this, the respective vector of the 4velocity is

$$(u^\mu) = \begin{pmatrix} h \\ \mathbf{0} \end{pmatrix} \quad , \tag{6}$$

$h = 1/f$. This coincides with Tolman's eqn (95.9). u^μ is not a tangent vector of a geodesic, as a comparison with Tolman's third eqn (144.4) reveals. In my notation, this said equation of a geodesic reads $u^0 = h^2 F$, F being a constant of integration, which is in contradiction to eqn (6). That is to say, a particle at rest in the static co-ordinate frame is not a freely falling one.

This also becomes obvious, if I calculate the 4acceleration w^μ to which such a test particle is subjected:—

$$\begin{aligned} w^\mu &= \frac{Du^\mu}{ds} \\ &= \frac{du^\mu}{ds} + \Gamma^\mu_{\varrho\sigma} u^\varrho u^\sigma \\ &= \frac{\partial u^\mu}{\partial x^\varrho} u^\varrho + \Gamma^\mu_{\varrho\sigma} u^\varrho u^\sigma \quad . \end{aligned}$$

The first summand vanishes because u^0 depends only on r and all components u^i are zero. This implies

$$w^\mu = \Gamma^\mu_{00} (u^0)^2 = \Gamma^\mu_{00} h^2 \quad .$$

The Christoffel symbols Γ^μ_{00} associated with the static metric are, according to Tolman (p. 250),

$$\begin{aligned} \Gamma^1_{00} &= \frac{1}{2} f^4 \frac{d \ln f^2}{dr} = \frac{1}{2} f^2 \frac{df^2}{dr} \\ &= -\frac{r}{a^2} f^2 \quad , \end{aligned}$$

and

$$\Gamma^0_{00} = \frac{d \ln f}{dt} = 0 \quad .$$

The only non-vanishing component of w^μ, therefore, is

$$w^1 = -\frac{r}{a^2} \quad . \tag{7}$$

This means that on a resting particle there acts a constant Newtonian force of this magnitude per unit mass towards the origin $r = 0$.

2.2 Robertson-Walker Co-Ordinates

The equations of transformation from static (not overlined) to R. W. co-ordinates (overlined) are

$$\bar{u}^{\mu} = u^{\varrho}\,\frac{\partial \bar{x}^{\mu}}{\partial x^{\varrho}} \quad . \tag{8}$$

This leads to

$$\begin{aligned}
\bar{u}^{0} &= u^{0}\,\frac{\partial \bar{x}^{0}}{\partial x^{0}} = u^{0}\,\frac{\partial \bar{t}}{\partial t} \\
&\overset{(5),(6)}{=} h(r[\bar{t},\bar{r}]) = \frac{1}{\sqrt{1-\dfrac{\chi^{2}}{a^{2}}\,\exp\left(\dfrac{2c\bar{t}}{a}\right)}} \quad , \\
\bar{u}^{1} &= u^{0}\,\frac{\partial \bar{x}^{1}}{\partial x^{0}} = \frac{u^{0}}{c}\,\frac{\partial \chi}{\partial t} \\
&\overset{(5),(6)}{=} -h^{2}(r[\bar{t},\bar{r}])\,\frac{r}{a}\,e^{-\frac{ct}{a}} = -\frac{\chi}{a}\,\frac{1}{\sqrt{1-\dfrac{\chi^{2}}{a^{2}}\,\exp\left(\dfrac{2c\bar{t}}{a}\right)}} \quad ,
\end{aligned}$$

or

$$(\bar{u}^{\mu}) = \frac{1}{\sqrt{1-\dfrac{\chi^{2}}{a^{2}}\,\exp\left(\dfrac{2c\bar{t}}{a}\right)}}\begin{pmatrix} 1 \\ -\dfrac{\chi}{a} \\ 0 \\ 0 \end{pmatrix} \quad . \tag{9}$$

In R. W. co-ordinates, therefore, the particle has a spatial velocity that is directed towards the origin $\chi = 0$. At the origin itself $\bar{u}^{1} = 0$.

The 4acceleration can likewise be transformed into R. W. co-ordinates by application of eqn (8):—

$$\bar{w}^{\mu} = w^{\varrho}\,\frac{\partial \bar{x}^{\mu}}{\partial x^{\varrho}} \quad .$$

As only $w^{1} \neq 0$, I get:—

$$\begin{aligned}
\bar{w}^{0} &= c\,\frac{\partial \bar{t}}{\partial r}\,w^{1} \\
&\overset{(5)}{=} \frac{1}{2}\,a\,\frac{d\ln f^{2}}{dr}\,w^{1} = \frac{\dfrac{r^{2}}{a^{3}}}{1-\dfrac{r^{2}}{a^{2}}} \quad .
\end{aligned}$$

Exploiting now prop. 1:—

$$\bar{w}^0 = \frac{\dfrac{\chi^2}{a^3}\,\exp\left(\dfrac{2c\bar{t}}{a}\right)}{1-\dfrac{\chi^2}{a^2}\,\exp\left(\dfrac{2c\bar{t}}{a}\right)} \quad .$$

In the same manner I proceed to obtain

$$\begin{aligned}
\bar{w}^1 &= \frac{\partial\chi}{\partial r}\,w^1 \\
&\overset{(5)}{=} \left(h + r\frac{dh}{dr}\right) e^{-\frac{ct}{a}}\,w^1 \;=\; h\left(1-\frac{r}{2}\,\frac{d\ln f^2}{dr}\right) e^{-\frac{ct}{a}}\,w^1 \\
&= h\left(1+\frac{r^2}{a^2}\,\frac{1}{1-\dfrac{r^2}{a^2}}\right) e^{-\frac{ct}{a}}\,w^1 \\
&= -\frac{h\,e^{-\frac{ct}{a}}}{1-\dfrac{r^2}{a^2}}\cdot\frac{r}{a^2} \\
&\overset{\substack{(5),\\ (\text{prop. } 1)}}{=} -\frac{\dfrac{\chi}{a^2}}{1-\dfrac{\chi^2}{a^2}\,\exp\left(\dfrac{2c\bar{t}}{a}\right)} \quad .
\end{aligned}$$

Cast into one vector equation:—

$$(\bar{w}^\mu) = \frac{\dfrac{\chi}{a^2}}{1-\dfrac{\chi^2}{a^2}\,\exp\left(\dfrac{2c\bar{t}}{a}\right)} \begin{pmatrix} \dfrac{\chi}{a}\,\exp\left(\dfrac{2c\bar{t}}{a}\right) \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad . \tag{10}$$

This means that the acceleration of a particle that is at rest in static co-ordinates undergoes an accelerated motion in R. W. coordinates: As both $\bar{u}^i$ and $\bar{w}^i$ are negative, this motion is directed and accelerated towards the origin $\chi = 0$. This particle does not travel along a geodesic, which is indicated by the fact that $w^\mu \neq 0$ in any co-ordinate system.

3 Four-Velocity of a Particle that is Falling Freely in the Robertson-Walker Co-Ordinates

3.1 Transformation into Static Co-Ordinates

The 4velocity of freely falling particles the 4velocity of which is

$$(\bar{u}^\mu) = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}$$

in R. W. co-ordinates will now be transformed into static co-ordinates:—

The equations of transformation are

$$u^\mu = \bar{u}^0 \, \frac{1}{c} \, \frac{\partial x^\mu}{\partial \bar{t}} \quad ,$$

where the bar again denotes the R. W. co-ordinates. For the non-vanishing components I get

$$\begin{aligned}
u^0 \quad &= \quad \bar{u}^0 \, \frac{\partial t}{\partial \bar{t}} \\
&\overset{\text{(prop. 1)}}{=} \quad 1 + \frac{\frac{\chi^2}{a^2} \exp\left(\frac{2c\bar{t}}{a}\right)}{1 - \frac{\chi^2}{a^2} \exp\left(\frac{2c\bar{t}}{a}\right)} \\
&\overset{\text{(prop. 1)}}{=} \quad 1 + \frac{\frac{r^2}{a^2}}{1 - \frac{r^2}{a^2}} \\
&= \quad h^2 \quad . \\
u^1 \quad &= \quad \bar{u}^0 \, \frac{1}{c} \, \frac{\partial r}{\partial \bar{t}} \\
&\overset{\text{(prop. 1)}}{=} \quad \frac{\chi}{a} \, e^{\frac{c\bar{t}}{a}} \\
&\overset{\text{(prop. 1)}}{=} \quad \frac{r}{a} \quad .
\end{aligned}$$

The two final equation can be combined to the vector equation

$$(u^\mu) = \begin{pmatrix} h^2 \\ \frac{r}{a} \\ 0 \\ 0 \end{pmatrix} \quad . \tag{11}$$

3.2 The World-Line $(\bar{u}^\mu) = (1, \mathbf{0})^T$ is a Geodesic

Proposition 2. *The 4velocity* $(\bar{u}^\mu) = (1, \mathbf{0})^T$ *in R. W. co-ordinates is the tangent vector of a geodesic curve.*

Proof. The equations of a geodesic are $w^\mu = 0$ or

$$u^{\mu\prime} + \Gamma^\mu_{\varrho\sigma} u^\varrho u^\sigma = 0 \quad ,$$

where the accent denotes the derivative with respect to s.[‡] On inserting $(\bar{u}^\mu) = (1, \mathbf{0})^T$ in the R. W. co-ordinates, the l. h. s. becomes $\overline{\Gamma}^\mu_{00}$. But $\overline{\Gamma}^\mu_{00} = 0$ is identically satisfied in the R. W. M.

If a vector is a tangent vector of a geodesic in one particular co-ordinate system, it has this property in every other. □

It is illuminating, however, to take a look at this 4velocity from the point of view of static co-ordinates:—

According to Tolman, p. 351, the 4velocity of a particle that moves along a geodesic, or the tangent vector of this curve, in the static metric is given by

$$\begin{aligned} c\,t' &= F h^2 = u^0 \quad , \\ r' &= \pm\sqrt{F^2 - f^2} = u^1 \quad , \end{aligned}$$

if the angular momentum of the particle is zero. F is a constant of integration: the energy constant.

From eqn (11) I take $u^0 = h^2$ and $u^1 = r/a$ for a freely falling particle, which means that it follows a geodesic curve along which the energy constant is $F = 1$ in the static metric.

4 Visualisation in Three Dimensions

4.1 dS^2 as a Hypersurface in the Minkowski Space-Time

I now consider the two-dimensional de Sitter space-time dS^2 as a hypersurface immersed in the three-dimensional Minkowski space-time $\mathbb{M}^3$. The cartesian co-ordinates of the latter be x^0, x^1, and x^4, so that the line element within $\mathbb{M}^3$ is

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^4)^2 \ .$$

The de Sitter space-time dS^2 is then defined by the constraint

$$dS^2 = \left\{ x \in \mathbb{M}^3 \middle| \ (x^0)^2 - (x^1)^2 - (x^4)^2 = -a^2 \right\} \ . \tag{12}$$

This is a hyperboloid of one sheet.

[‡]Throughout this text I will indicate the derivative with respect to the arc length by an accent.

4.2 Freely Falling Body in Planar Co-Ordinates

The constraint of eqn (12) is fulfilled, if the *planar co-ordinates* $\bar{t}, \chi$ are introduced:

$$\begin{aligned} x^0 &= a \,\mathrm{sh}\,\frac{c\bar{t}}{a} + \frac{1}{2a}\chi^2 e^{\frac{c\bar{t}}{a}} , \\ x^1 &= \chi e^{\frac{c\bar{t}}{a}} , \\ x^4 &= a \,\mathrm{ch}\,\frac{c\bar{t}}{a} - \frac{1}{2a}\chi^2 e^{\frac{c\bar{t}}{a}} . \end{aligned} \tag{13}$$

$\bar{t}$ and χ thus provide a possible parametrisation of dS^2. The line element ds acquires the Robertson-Walker form of eqs (3) and (4). The geodesic world lines of freely falling particles the "4velocity"[§] of which is $(\bar{u}^\mu) = (1,0)^T = (c\,\bar{t}', \chi')^T$ are obtained, if χ is held constant and $\bar{t}$ varied from $-\infty$ to $+\infty$. These are sketched in fig. 1.

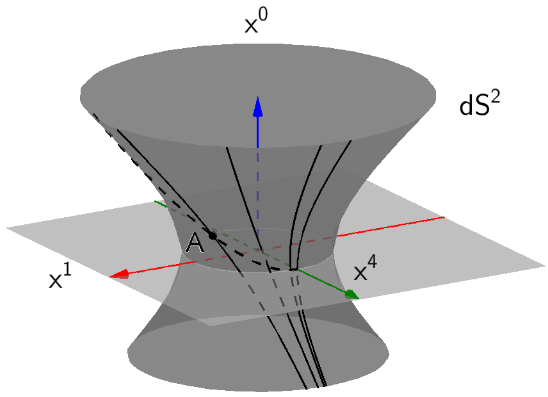

Figure 1: The two-dimensional de Sitter space-time is a hyperboloid of one sheet. The metric, however, is Minkowskian, not Euclidean. The geodesics followed by freely falling particles the "4velocity" of which, in R. W. co-ordinates, is $(\bar{u})^0 = (1,0)^T$ are shown as solid lines. The radial co-ordinate χ, constant along each of these geodesics, increases from right to left, running from $\chi = 0$ to $\chi = a$. The dashed curve is the isochrone $\bar{t} = 0$. It intersects the geodesic $\chi = a$ at $A\left(\frac{a}{2} \,\middle|\, a \,\middle|\, \frac{a}{2}\right)$.

The Difference between a Curve on the Manifold and its Image on a Chart.— A fictitious observer who lives within dS^2 would never see these geodesics as they appear from a point of view in the ambient space, detached from the manifold dS^2, as sketched in fig. 1. Instead he would be confined to his *chart*: to the co-ordinate plane spanned by the axes of $\bar{t}$ and χ. In this chart these geodesics appear as simple

[§]In dS^2 there are only 2 co-ordinates: $\bar{t}$ and χ. For this reason I use "4velocity" in quotation marks when referring to dS^2.

lines parallel to the $\bar{t}$-axis. These are the world-lines of bodies which are at rest in a co-expanding frame of reference, see fig. 2.

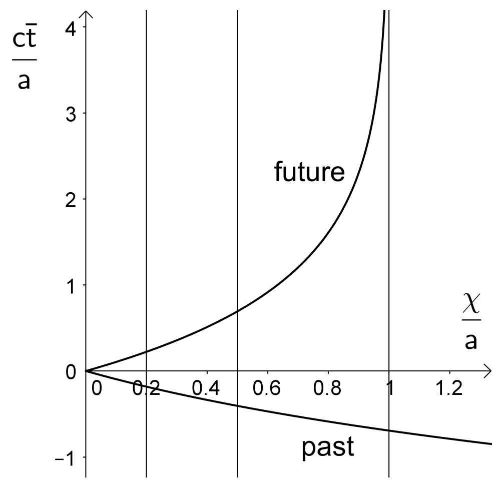

Figure 2: Word-lines of freely falling bodies the 4velocity of which is $(\bar{u}^\mu) = (1,0)^T$. These images of the geodesics sketched in fig. 1 are just lines parallel to the $\bar{t}$-axis, because each of these bodies remains at its constant (co-expanding) spatial position χ. Also shown are the future and the past light-cones of an observer at $\bar{t} = 0$ and $\chi = 0$. The former reveals an event horizon: The future light-cone will intersect only world-lines of such freely co-expanding bodies as are closer than $\chi = a$. This betrays the "event horizon" inherent to the de Sitter world:—The observer at $\chi = 0$ cannot influence any part of the universe that lies further away than $\chi = a$ on the co-moving scale. On the other hand, there is no "particle horizon":— Unlike in the Friedmann-Lemaître models, the past light-cone is not confined to the region $\chi \in [0, \approx c/H_0]$. All parts of the universe are visible to the chosen observer. See also Sect. 4.3.2 and fig. 3.

4.3 Event Horizon in the de Sitter Universe

4.3.1 Chart $\bar{t}$, χ: Co-Expanding Co-Ordinates

Proposition 3 (Future light-cone)**.** *I consider an observer at* $\bar{t} = 0$ *and* $\chi = 0$*, who sends out a light-signal. After the time* $\bar{t}$ *has elapsed, it has travelled the distance*

$$\chi = a\left(1 - e^{-\frac{c\bar{t}}{a}}\right) = \chi_f(\bar{t}) \quad , \ \bar{t} > 0\,. \tag{14}$$

The curve $(\chi, \bar{t})^T = (\chi_f[\bar{t}], \bar{t})^T$ is the future light-cone of the observer at the origin.

Proof of Prop. 3. Along a light-path, $ds = 0$, and

$$R(\bar{t})\, d\chi = c\, d\bar{t}$$

results. On inserting the r. h. s. of eqn (4), this becomes

$$\begin{aligned}\chi &= c \int_0^{\bar{t}} e^{-\frac{c\,t}{a}}\, dt \\ &= a\left(1 - e^{-\frac{c\,\bar{t}}{a}}\right) \quad . \end{aligned}$$ □

This converges to the finite value a, if $\bar{t} \to \infty$. This implies

Proposition 4 (Event horizon)**.** *An observer at* $\chi = 0$ *can only influence the region* $\chi \in [0, a[$.

The future light-cone is sketched in fig. 2. The event horizon is evident by the line $\chi = a$, which is an asymptote to this light-cone.

Past Light-Cone.—This light-cone is given by the integral

$$\begin{aligned}\chi_{\mathrm{P}}(\bar{t}) &= c \int_{\bar{t}}^{0} e^{-\frac{c\,t}{a}}\, dt \\ &= a\left(e^{-\frac{c\,\bar{t}}{a}} - 1\right) \quad , \ \bar{t} < 0\,. \end{aligned} \tag{15}$$

This diverges for $\bar{t} \to -\infty$, indicating that there is no "particle horizon": the whole universe is visible to the observer. It will become evident in the next subsection, however, that no object possesses an *emission distance* in excess of $R(\bar{t})\chi = a$. The past light-cone is sketched in fig. 2.

4.3.2 World-Lines in a Time–Radial Distance Diagram

The radial distance of an event at time $\bar{t} < 0$ from an observer at $\chi = 0$ and $\bar{t} = 0$ is $D_\chi = R(\bar{t}) \cdot \chi$. If this event lies on this observer's light-cone, D_χ will be called the "*emission distance*" to this event. It means the radial distance in which the respective objects stood from the observer *at the time it emitted the signal that he now, i. e. at time* 0*, receives.* These co-ordinates were used in fig. 3. Here, the past light-cone is given by the curve

$$\begin{pmatrix} R(\bar{t})\chi \\ \bar{t} \end{pmatrix} = \begin{pmatrix} \exp\left(\frac{c\,\bar{t}}{a}\right) \chi_{\mathrm{P}}(\bar{t}) \\ \bar{t} \end{pmatrix} \quad ,$$

with $\chi_{\mathrm{P}}(\bar{t})$ from eqn (15).

In this kind of representation, the event horizon is immediately obvious in the past light-cone: as seen by an observer who is at $\bar{t} = 0$ at $\chi = 0$ there is no object the emission distance of which exceeds a. *This observer cannot have any information of the universe beyond the distance a.*

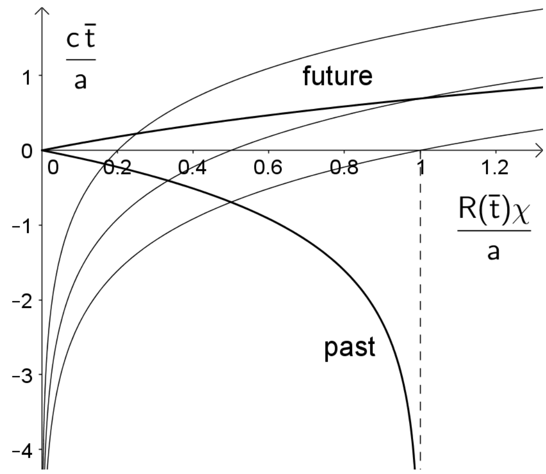

Figure 3: Same as fig. 2, but with the time-dependent radial distance $R(\bar{t})\chi$ shown on the abscissa. The world-lines of the co-expanding bodies are no longer parallel to the ordinate. Their points of intersection with the past light-cone mark their respective emission distances. As the past light-cone, in this representation, is confined to $R\chi \in [0, a[$, there is no object at an emission distance greater than a, which means that an observer at $\chi = 0$ and $\bar{t} = 0$ can have no information from distances in excess of a. This indicates the event horizon. The future light-cone appears to be unlimited, however, as was shown in Sect. 4.3.1 and displayed in fig. 2, no region the co-moving distance χ of which exceeds a can be reached by a signal from the origin.

4.3.3 The Event Horizon

The—well known—results from the two previous sections can be summarised in

Proposition 5 (Event horizon at a)**.** *In the de Sitter universe there exists an event horizon in the sense that an observer who rests at the origin at time 0 cannot receive any information from regions of the universe the emission distances of which exceed a, nor can he influence any region of the universe the co-moving distance of which exceeds this value.*

In an expanding universe, the term "radial distance" has no meaning unless the information is added, *to which instant of time* an indicated value of the distance refers. Bearing this in mind, I can re-state prop. 5:—

Proposition 6 (Corollary)**.**

1. *The furthest point to which a signal can be sent from the origin lies at the distance a* at that very moment the signal is sent.
2. *The furthest object from which a signal can be received at the origin was at the distance a* at the very moment the signal was sent.

Fig. 4 displays the situation on the manifold dS^2 itself. Shown are one branch of both the future and past light-cones of an observer at $\bar{t} = 0$, $\chi = 0$ as well as the geodesics of freely falling bodies that co-expand with the universe.

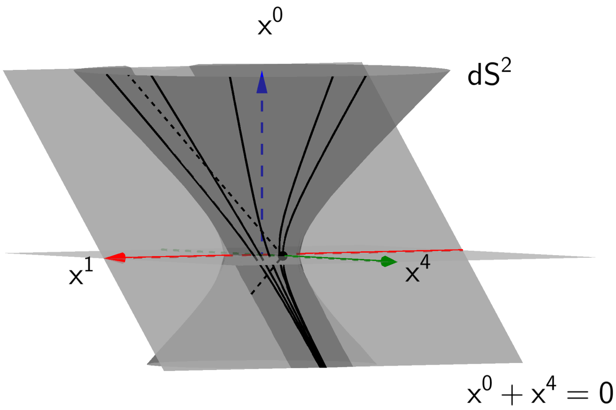

Figure 4: The event horizon as it manifests itself on the manifold dS^2. The dotted lines represent one branch of the future and past light-cones of an observer at $\bar{t} = 0$, $\chi = 0$. Solid lines are the trajectories of freely falling bodies with $(\bar{u}^\mu) = (1, 0)^T$, with χ increasing from 0 (rightmost curve) to a. The geodesic along which $\chi = a$ is not reached by the observer's light-cone, hence he cannot influence any point beyond this co-moving distance. The past light-cone terminates at the line of intersection of dS^2 with the plane $x^0 + x^4 = 0$, where the end of the chart provided by the planar co-ordinates is reached. This limiting plane is also shown in the figure. There is no particle horizon: all these geodesics intersect the past light-cone.

Proposition 7 (Images of the light-cones on dS^2)**.** *The future and past light-cones of an observer at $\bar{t} = 0$, $\chi = 0$ map to the curves*

$$\frac{\mathbf{r}}{a} = \begin{pmatrix} e^{\frac{c\bar{t}}{a}} - 1 \\ e^{\frac{c\bar{t}}{a}} - 1 \\ 1 \end{pmatrix} \quad , \ \bar{t} > 0 \quad (future)$$

and

$$\frac{\mathbf{r}}{a} = \begin{pmatrix} e^{\frac{c\bar{t}}{a}} - 1 \\ 1 - e^{\frac{c\bar{t}}{a}} \\ 1 \end{pmatrix} \quad , \ \bar{t} < 0 \quad (past)$$

on dS^2. $\mathbf{r} = (x^0, x^1, x^4)^T$ is the position vector in the ambient space-time $\mathbb{M}^3$.

The past light-cone obviously terminates at the point $(-a|\ a\ |a)$*, if* $t \to -\infty$.

Proof. I insert the expressions (14) and (15) into the planar co-ordinates given in eqs (13). Throughout this proof, I will drop the bars over t:—

$$\begin{aligned}
\frac{x^0}{a} &= \operatorname{sh}\frac{c\,t}{a} + \frac{1}{2}\left(1 - e^{-\frac{c\,t}{a}}\right)^2 e^{\frac{c\,t}{a}} \ , \\
\frac{x^1}{a} &= \pm\left(1 - e^{-\frac{c\,t}{a}}\right) e^{\frac{c\,t}{a}} \ , \\
\frac{x^4}{a} &= \operatorname{ch}\frac{c\,t}{a} - \frac{1}{2}\left(1 - e^{-\frac{c\,t}{a}}\right)^2 e^{\frac{c\,t}{a}} \ ,
\end{aligned}$$

where the plus sign relates to the future, the minus sign to the past light-cone. Further algebra gives:—

$$\begin{aligned}
\frac{x^0}{a} &= \operatorname{sh}\frac{c\,t}{a} + \frac{1}{2}\left(e^{\frac{c\,t}{2a}} - e^{-\frac{c\,t}{2a}}\right)^2 \ , \\
\frac{x^1}{a} &= \pm\left(e^{\frac{c\,t}{a}} - 1\right) \ , \\
\frac{x^4}{a} &= \operatorname{ch}\frac{c\,t}{a} - \frac{1}{2}\left(e^{\frac{c\,t}{2a}} - e^{-\frac{c\,t}{2a}}\right)^2 \ .
\end{aligned}$$

$$\begin{aligned}
\frac{x^0}{a} &= \operatorname{sh}\frac{c\,t}{a} + 2\operatorname{sh}^2\frac{c\,t}{2a} \ , \\
\frac{x^1}{a} &= \pm\left(e^{\frac{c\,t}{a}} - 1\right) \ , \\
\frac{x^4}{a} &= \operatorname{ch}\frac{c\,t}{a} - 2\operatorname{sh}^2\frac{c\,t}{2a} \ .
\end{aligned}$$

$$\begin{aligned}
\frac{x^0}{a} &= \operatorname{sh}\frac{c\,t}{a} + \operatorname{ch}\frac{c\,t}{a} - 1 = e^{\frac{c\,t}{a}} - 1 \ , \\
\frac{x^1}{a} &= \pm\left(e^{\frac{c\,t}{a}} - 1\right) \ , \\
\frac{x^4}{a} &= \operatorname{ch}\frac{c\,t}{a} - \operatorname{ch}\frac{c\,t}{a} + 1 \equiv 1 \ .
\end{aligned}$$

□

4.4 Static Co-Ordinates

Another possible set of co-ordinates that fulfil the constraint of eqn (12) are the *static co-ordinates* t, r:

$$\begin{aligned} x^0 &= a\,f\ \mathrm{sh}\,\frac{ct}{a}\ , \\ x^1 &= r\ , \\ x^4 &= a\,f\ \mathrm{ch}\,\frac{ct}{a}\ , \end{aligned} \tag{16}$$

with

$$f = \sqrt{1 - \frac{r^2}{a^2}}\quad ,$$

see eqn (2). The event horizon is obvious from the zero of f at $r = a$. The line element takes the form of eqn (1). Between the planar and the static co-ordinates the equations of transformation of prop. 1 and eqs (5) obtain. The parameter lines t = const. (dashed) and r = const. (dotted) are shown in fig. 5.

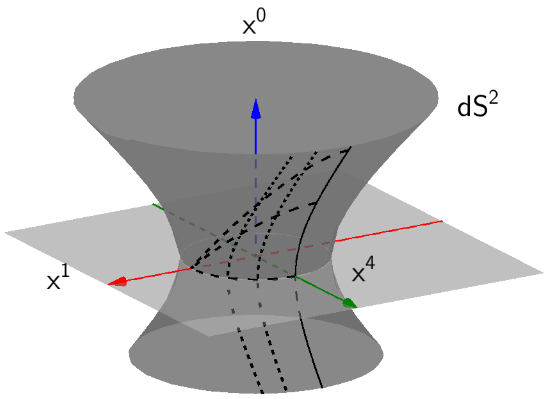

Figure 5: Parameter lines r = const. (dotted) and t = const. (dashed) of the static co-ordinates are shown. The former are no geodesics, with the exception of $r = 0$, which is shown as a solid curve (see Sect. 2.1).

4.5 A Freely Falling Body in Static Co-Ordinates

Again I consider test particles that fall freely along world lines which are characterised by the "4velocity" $(\bar{u}^\mu) = (1, 0)^T$ in R. W. co-ordinates. As I showed in Sect. 3.1, this "4velocity", expressed in static co-ordinates, is

$$(u^\mu) = \begin{pmatrix} h^2 \\ \frac{r}{a} \end{pmatrix} = \begin{pmatrix} ct' \\ r' \end{pmatrix}\quad , \tag{11}$$

where the accent denotes the derivative with respect to s, and only the two co-ordinates t and r are considered. On re-parametrising from s to t along the geodesics, I get

$$\begin{aligned}\frac{dr}{dt} &= \frac{r'}{t'} \\ &= \frac{r}{a}\cdot\frac{c}{h^2}\end{aligned}$$

or

$$\frac{dr}{dt} = c\,\frac{r}{a}\left(1-\frac{r^2}{a^2}\right) \quad ,$$

which can easily be integrated to give

$$\begin{aligned} c\,t(r) &= \int \frac{dr}{\frac{r}{a}\left(1-\frac{r^2}{a^2}\right)} \\ &= \frac{a}{2}\ln\frac{\frac{r^2}{a^2}}{1-\frac{r^2}{a^2}} + A \\ &= a\,\ln h\,\frac{r}{a} + A \quad , \end{aligned} \tag{17}$$

where A is a constant of integration.

On varying r from 0 to a, t will take all values from $-\infty$ to $+\infty$, which indicates that the "de Sitter horizon" at $r = a$ cannot be reached by any material body from the origin within a finite interval of co-ordinate time t.

The geodesics $(x^0, x^1, x^4)^T = (x^0[r], x^1[r], x^4[r])^T$ of a freely falling observer in dS^2 are obtained on expressing $c\,t$ in eqs (16) by the function from the last eqn (17). By so doing, r is utilised as their parameter. These geodesics are sketched in fig. 6. Of course, they are the same ones as those of fig. 1, however, they terminate at the de Sitter horizon $r = a$, where the chart that is provided by the static co-ordinates ends. This horizon coincides with the curve of intersection of dS^2 with the plane $-x^0 + x^4 = 0$ (see fig. 6).

These geodesics form a one-parameter family of curves. The distinguishing parameter is the constant of integration in eqn (17). The geodesic on which $A = 0$, or, in planar co-ordinates, $\chi = a$, terminates at the point $A\left(\frac{a}{2}\,\middle|\,a\,\middle|\,\frac{a}{2}\right)$ on the de Sitter horizon $r = a$. In planar co-ordinates, this is its point of intersection with the isochrone $\bar{t} = 0$, see fig. 1.

The Images of the Geodesics on the Chart.—As I did in connection with the planar co-ordinates, it is worthwhile to compare the geodesics on the manifold dS^2, shown in figs 1 and 6, with their images on the chart, their world-lines, which

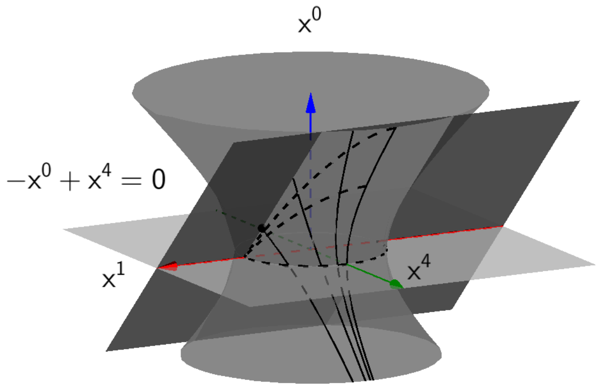

Figure 6: Geodesics of freely falling particles (solid lines) the "4velocity" of which, in R. W. co-ordinates, is $(\bar{u}^\mu) = (1, 0)^T$. This drawing is based on the chart provided by the static co-ordinates. On the geodesics the constant of integration A increases from $A = 0$ (leftmost curve) to $A \to \infty$ (rightmost curve). This corresponds to $\chi = a$ and $\chi = 0$, resp., in planar co-ordinates. All these curves terminate at the de Sitter horizon $r = a$, where the chart provided by the static co-ordinates ends. The—marked—termination point of the leftmost curve ($A = 0$ or $\chi = a$) is $A\left(\frac{a}{2} \,\middle|\, a \,\middle|\, \frac{a}{2}\right)$. The de Sitter horizon is the curve of intersection of dS^2 with the plane $-x^0 + x^4 = 0$. Also shown are three isochrones for $t = \text{const.} \geqslant 0$, $r \geqslant 0$ (dashed). The isochrone $t = 0$, however, has been extended to a semi-circle.

are given in the functional form of $c\,t = c\,t(r)$ in eqn (17). They are shown in fig. 7. Most striking is this difference between the geodesics on the manifold dS^2 and their images on the chart of static co-ordinates:—

The former can only be parametrised as far as their respective termination points, where they seem to end. The latter, their respective world-lines, however, do not display such end points. Instead they merge with the asymptote $r = a$ and extend to ∞.

An equation for the light-cones of a fictitious observer at $t = 0$, $r = 0$ can be written down:

Proposition 8 (Light-cones of an observer at the origin)**.** *The light-cones of an observer at $t = 0$, $r = 0$ are given by*

$$c\,t = \pm a \operatorname{arth} \frac{r}{a}\,, \tag{18}$$

where the plus sign relates to the future, the minus sign to the past light-cone.

Proof. Along a light-path, $ds = 0$, which yields

$$f\,c\,dt = h\,dr$$

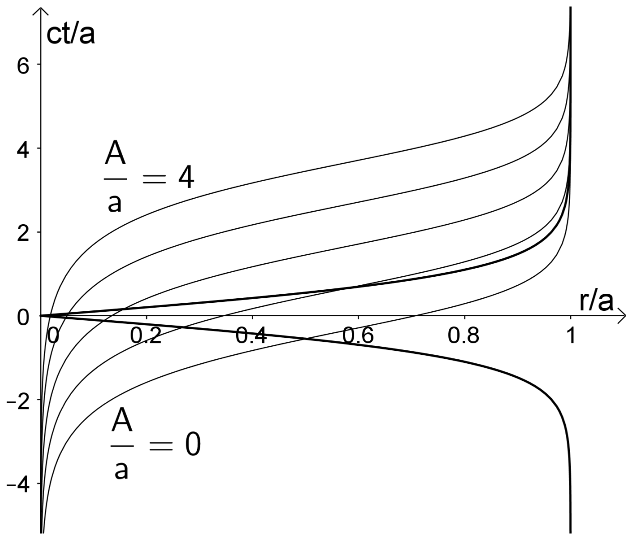

Figure 7: World-lines of freely falling bodies that follow the geodesics sketched in fig. 6, as they appear on the map provided by the static co-ordinates. In the language of differential geometry, these are the images of the geodesics on the manifold, as they map onto this chart. A is the constant of integration from eqn (17), which discriminates between the curves. The values 0, 1, 2, 3, 4 have been chosen for A/a. There is an asymptote at $r = a$: no point on dS^2 that lies beyond the image of this line on the manifold can be represented on this chart. Note that, unlike their image curves on dS^2, these world-lines do not display any apparent termination points. Also shown (thick curves) are future and past light-cones of an observer at the origin:—No region beyond $r = a$ can be influenced by him, nor can he be influenced by anything further away than $r = a$.

or

$$c\,dt = h^2\,dr\,.$$

On the future light-cone:

$$c\,t = \int_0^r \frac{dr}{1 - \frac{r^2}{a^2}} = a \text{ arth} \frac{r}{a} \quad .$$

On the past light-cone:

$$c\,t = \int_r^0 \frac{dr}{1 - \frac{r^2}{a^2}} = -a \text{ arth} \frac{r}{a} \quad .$$

In both cases $r \geqslant 0$. □

Owing to the fact that the chart of dS^2 provided by these co-ordinates maps a considerably smaller fraction of the manifold than does the one provided by the planar, the most distant freely falling body that can be influenced by the observer resting at the origin is not the one that follows the trajectory $A = 0$ or $\chi = a$, as in the case of planar co-ordinates:—

Proposition 9. *In static co-ordinates, the future light-cone of a fictitious observer at $t = 0$, $r = 0$, will intersect the trajectory of a freely falling body with $\bar{u}^\mu = (1, 0)^T$ (in planar co-ordinates) only as long as $A > a \ln 2$.*

With such bodies as have $A \in [0, a \ln 2]$ a light-signal sent out from $r = 0$ at $t = 0$ will not catch up before they reach $r = a$. This can be very clearly seen in fig. 8.

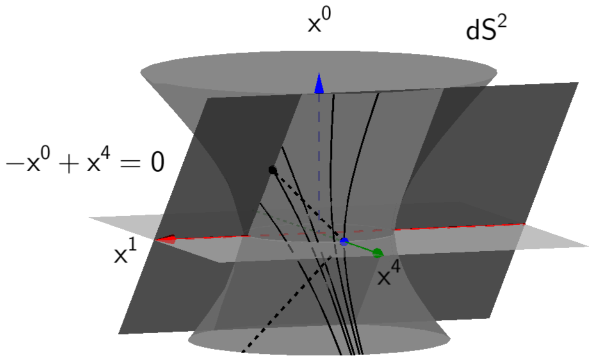

Figure 8: Geodesics of freely falling particles (solid lines) the "4velocity" of which, in R. W. co-ordinates, is $(\bar{u}^\mu) = (1, 0)^T$. On the geodesics the constant of integration A increases from $A = 0$ (leftmost curve) to $A \to \infty$ (rightmost curve). This corresponds to $\chi = a$ and $\chi = 0$, resp., in planar co-ordinates, just as in fig. 6. Also shown are the images of respectively one branch of the future and past light-cone of an observer at $t = 0$, $r = 0$ in the chart, which is $(0 | 0 | 1)$ on dS^2. The most distant body the trajectory of which intersects that of the forward light-signal from the observer on this side of the line of termination of the chart is the one for which $A = a \ln 2$, as shown in prop. 9. The point of intersection is $(a | a | a)$, see prop. 10.

Proof of prop. 9. The equation of the geodesics in question (eqn [17]) can be rewritten as

$$\frac{\frac{r}{a}}{\sqrt{1 - \frac{r^2}{a^2}}} = e^{\frac{ct - A}{a}} \; . \tag{19}$$

On the other hand,

$$\frac{r}{a} = \text{th}\,\frac{c\,t}{a}$$

on the future light-cone of an observer at $t = 0$, $r = 0$. Observing that

$$\frac{\text{th}\,x}{\sqrt{1 - \text{th}^2\,x}} = \text{sh}\,x\,,$$

eqn (19) becomes

$$\text{sh}\,\frac{c\,t}{a} = e^{\frac{c\,t-A}{a}}$$

or

$$\tfrac{1}{2}\,e^{\frac{c\,t}{a}} - \tfrac{1}{2}\,e^{-\frac{c\,t}{a}} = e^{\frac{c\,t-A}{a}}\,,$$

from which

$$1 - e^{-2\frac{c\,t}{a}} = 2e^{-\frac{A}{a}}$$

and

$$e^{-2\frac{c\,t}{a}} = 1 - 2e^{-\frac{A}{a}}$$

follow. The last equation, however, possesses a solution in t only as long as its r. h. s. is positive, which means that

$$e^{\frac{A}{a}} > 2$$

or $A > a\,\ln 2$. □

Proposition 10 (Point at which the future light-cone reaches the horizon)**.** *The future light-cone of an observer at $t = 0$, $r = 0$ reaches the horizon $r = a$ at the point $(a|\,a\,|a)$.*

Proof. I begin by writing down the definition of the static co-ordinates:—

$$\begin{aligned} x^0 &= a\,f\,\text{sh}\,\frac{ct}{a}\,, \\ x^1 &= r\,, \\ x^4 &= a\,f\,\text{ch}\,\frac{ct}{a}\,. \end{aligned} \tag{16}$$

If $r = a$, $x^1 = a$, which is obvious. As to x^0 and x^4, I express both the sh and the ch by the th:—

$$\frac{x^{0|4}}{a} = \frac{f}{\sqrt{1 - \text{th}^2 \frac{c\,t}{a}}} \cdot \begin{cases} \text{th} \frac{c\,t}{a} \\ 1 \end{cases} .$$

Now I replace the argument $c\,t/a$ by its function of r/a on the future light-cone, see eqn (18):—

$$\frac{x^{0|4}}{a} = \frac{f}{\sqrt{1 - \frac{r^2}{a^2}}} \cdot \left\{ \begin{array}{c} \frac{r}{a} \\ 1 \end{array} \right\} = \left\{ \begin{array}{c} \frac{r}{a} \\ 1 \end{array} \right\} \xrightarrow[r \to a]{} 1 \quad .$$ □

4.6 The "Termination Points", beyond which the Geodesics cannot be traced in the Chart that is provided by the Static Co-Ordinates

The chart that is provided by the static co-ordinates ends at $r = a$. The geodesics, if parametrised by r according to eqn (17), seem to terminate here, as shown in figs 6 and 8. As the co-ordinate time t of the static co-ordinates becomes infinite at $r = a$, I will fall back on the planar co-ordinates $\bar{t}, \chi$ to circumvent this co-ordinate singularity in order to find the positions of the respective termination-points on dS^2.

Proposition 11 (Connection between planar and static co-ordinates along the geodesics)**.** *Given the geodesics followed by freely falling bodies the "4velocity" of which, in planar co-ordinates, is* $(\bar{u}^\mu) = (1, 0)^T$. *Be they parametrised in static co-ordinates according to eqn* (17)*, so that they form a one-parameter family of curves, the curve parameter of which is* $r \in [0, a[$*, and the "family parameter", which distinguishes between the particular geodesics, is* $A \in [0, \infty[$.

For the planar co-ordinates $\bar{t}$, χ *along one such geodesic the following obtains:—*

1. $c\,\bar{t} = a \ln \frac{r}{a} + A$ *(connection of curve parameters).*
2. $c\,\bar{t} \in]-\infty, A]$*, with* $c\,\bar{t} = A$ *at* $r = a$.
3. $\chi = a\ \exp(-A/a) = \text{const.}$ *(connection of "family parameters").*

There is no singularity at $r = a$ in the planar co-ordinates: the respective co-ordinate time $\bar{t}$ remains finite ($c\,\bar{t} = A$) there.

Proof of prop. 11. I take the equations of transformation (5):

$$\begin{aligned} c\,\bar{t} &= c\,t + a\ \ln f \quad , \\ \chi &= r\,h\,e^{-\frac{c\,t}{a}} \quad , \end{aligned} \tag{5}$$

and insert the r. h. s. of eqn (17) into the first of these equations:—

$$\begin{aligned} c\,\bar{t}(r) &= a \ln h \frac{r}{a} + A + a \ln f \\ &= a \ln \frac{r}{a} + A \;, \end{aligned} \tag{20}$$

with

$$f^2 = 1 - \frac{r^2}{a^2} = \frac{1}{h^2} \quad .$$

This proves No. 1.

No. 2 is proven by simply inserting into the second eqn (20) numbers for $r \in]0, a]$.

To prove No. 3, I plug the r. h. s. of eqn (17) into the second eqn (5):—

$$\begin{aligned} \chi &= r\,h\,\exp\left(-\ln h \frac{r}{a} - \frac{A}{a}\right) \\ &= \frac{r\,h}{h\frac{r}{a}}\, e^{-\frac{A}{a}} \\ &= a\, e^{-\frac{A}{a}} \quad . \end{aligned}$$

□

The respective point at which the particular geodesics fall off the chart and can no longer be traced by the static co-ordinates can also be found:—

Proposition 12 (Termination point of the geodesics in static co-ordinates)**.** *If parametrised in static co-ordinates according to eqn* (17)*, a given geodesic, distinguished by a given value of the "family parameter" A, will fall off the chart at the point*

$$\left(\frac{a}{2}\, e^{\frac{A}{a}} \,\middle|\, a \,\middle|\, \frac{a}{2}\, e^{\frac{A}{a}}\right) \quad .$$

Form this proposition, with reference to prop. 10, I find again that the future light-cone of an observer at $t = 0$, $r = 0$ intersects the geodesic on which $A = a \ln 2$ exactly at the horizon $r = a$:—

$$\left(\frac{a}{2}\, e^{\frac{A}{a}} \,\middle|\, a \,\middle|\, \frac{a}{2}\, e^{\frac{A}{a}}\right) = (a|\; a\;|a) \quad \Leftrightarrow \quad e^{\frac{A}{a}} = 2 \quad .$$

□

Proof of prop. 12. I insert $(c\bar{t}, \chi)^T = (A, a \exp[-A/a])^T$ into the r. h. s. of eqs (13):

$$\begin{aligned} x^0 &= a\,\mathrm{sh}\frac{A}{a} + \frac{1}{2}\, a\, e^{-2\frac{A}{a}}\, e^{\frac{A}{a}} \\ &= a\,\mathrm{sh}\frac{A}{a} + \frac{1}{2}\, a\, e^{-\frac{A}{a}} \\ &= \frac{a}{2}\, e^{\frac{A}{a}} \;. \end{aligned}$$

$$
\begin{aligned}
x^1 &= a\,e^{-\frac{A}{a}}\,e^{\frac{A}{a}} = a\,. \\
x^4 &= a\,\mathrm{ch}\,\frac{A}{a} - \frac{1}{2}\,a\,e^{-2\frac{A}{a}}\,e^{\frac{A}{a}} \\
&= a\,\mathrm{ch}\,\frac{A}{a} - \frac{1}{2}\,a\,e^{-\frac{A}{a}} \\
&= \frac{a}{2}\,e^{\frac{A}{a}}\,. \qquad \square
\end{aligned}
$$

4.7 The Limiting Geodesic for $A \to \infty$

The limit $A \to \infty$ cannot be carried out in the expression for the termination point of the geodesics that I wrote down in prop. 12. In oder to find out which geodesic is sorted out by this limit, I must go back to the planar co-ordinates.

According to prop. 11, $A \to \infty$ corresponds to $\chi = 0$. And on this curve, $\bar{t} \in]-\infty, +\infty[$. As $\chi = 0$ implies $r = 0$, $t = \bar{t}$, according to prop. 1, this curve can be traced all along even in static co-ordinates. That is to say, $A \to \infty$ refers to an observer at the origin. I have thus proven

Proposition 13 (Limiting curve for $A \to \infty$)**.** *If the limit $A \to \infty$ is performed in eqn* (17)*, the world-line of an observer who is at rest at the origin maps to the respective geodesic:*

$$
\begin{aligned}
x^0 &= a\ \mathrm{sh}\,\frac{c\,t}{a}\,, \\
x^1 &\equiv 0\,, \\
x^4 &= a\ \mathrm{ch}\,\frac{c\,t}{a}\,.
\end{aligned}
$$

5 Tangent Vectors and the Four-Velocity

5.1 Tangent Vectors of Manifolds without an Ambient Space

In the literature on differential geometry, I found two ways of defining the tangent vector of a curve, *if there is no ambient space* in which the manifold under consideration is immersed.

Equivalence class. Two curves the co-ordinates of which are $x^\mu(t)$ and $y^\mu(\tau)$ on a given chart are said to have the same tangent vector $\mathbf{t}$ at a point P, if

$$
\frac{dx^\mu}{dt} = \frac{dy^\mu}{d\tau} \quad \text{at } P.
$$

This definition is based on a chart or co-ordinate system from the beginning. Two curves are called *equivalent*, if they possess the same tangent vector. In this way, $\mathbf{t}$ is defined as the respective equivalence class.

If the length of arc, s, is the parameter, then, in a 4 dimensional space-time, the tangent vector is fully determined by the 4velocity

$$u^\mu = \frac{dx^\mu}{ds} \quad :$$

Two trajectories have the same tangent vector at a certain point, if their respective 4velocities, expressed in a chosen co-ordinate system, or chart, coincide there. In this sense, u^μ can be regarded as the tangent vector of the trajectory of a body.

Directional derivative. This definition is independent of any co-ordinate system:

A curve is regarded as a mapping γ of an interval $I \subset \mathbb{R}$ onto the manifold M. Given a function $f\colon\ M \to \mathbb{R}$, the directional derivative xf of f along γ is defined as

$$xf = \frac{d\,f \circ \gamma}{dt} \quad ,$$

where γ and f are assumed to be differentiable. $f \circ \gamma$ is, thus, a differentiable function that maps $\mathbb{R}$ onto itself. The directional derivative of f along γ is interpreted as the mapping $x\colon\ f \to \mathbb{R}$, with given curve γ, and this mapping by definition is regarded as the tangent vector of the curve γ.

The connection between these two definitions becomes obvious as soon as a chart $\varphi\colon\ M \to \mathbb{R}^n$ is introduced. Then $f \circ \gamma$ can be written as

$$f \circ \gamma = (f \circ \varphi^{-1}) \circ (\varphi \circ \gamma) \,. \tag{21}$$

$\varphi \circ \gamma = (x^0[t], \ldots, x^{n-1}[t])^T$ is a vector-valued function $\mathbb{R} \to \mathbb{R}^n$, and $f \circ \varphi^{-1} = \mathrm{f}(x^0, \ldots, x^{n-1})$ is a real-valued function of the variables $x^0, \ldots, x^{n-1}$. xf then takes the familiar form of gradient times tangent vector:—

$$xf = \frac{\partial \mathrm{f}}{\partial x^\mu}\,\dot{x}^\mu \,, \tag{22}$$

where the dot denotes the derivative with respect to t. After re-parametrisation from t to the arc-length s and specification on a 4 dimensional space-time, the components of the 4velocity are obviously the coefficients of the partial derivatives of f.

These partial derivatives are special tangent vectors in the following sense:—

At a given point on a differentiable manifold M, $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$ *denotes the tangent vector that assigns to the function* $f\colon\ M \to \mathbb{R}$ *the number*

$$\frac{\partial\, f \circ \varphi^{-1}}{\partial x^\mu} = \frac{\partial \mathrm{f}}{\partial x^\mu} \quad .$$

The ∂_μ form a basis of the tangent space at the point under consideration.

Summarising, I have

Proposition 14. *In a four-dimensional space-time, the components $u^\mu = x^{\mu\prime}$ of the four-velocity represent a tangent vector of a trajectory, expressed in the basis of the tangent space that is set up by the vectors ∂_μ at the respective point.*

5.2 Subspaces of $\mathbb{R}^m$

In the case that the n dimensional manifold in question is immersed in $\mathbb{R}^m$, $m > n$, the notion of a tangent vector can be adopted from the theory of surfaces. In this connection, the symbol $\mathbb{R}^m$ does not have any implication on the metric of the ambient space around the manifold. It does not necessarily have to be the Euclidean but could also be the Minkowskian metric, as it is the case with the de Sitter space-time, which is immersed in $\mathbb{M}^m$.

Every point on the manifold can be specified by its position vector in $\mathbb{R}^m$

$$\mathbf{r} = \begin{pmatrix} x^0 \\ \vdots \\ x^{m-1} \end{pmatrix} \quad .$$

As there exist one or more constrains of the form $F(x^0, \ldots, x^{m-1}) = 0$ between the co-ordinates x^A of $\mathbb{R}^m$ to define the manifold, there are only $n < m$ independent co-ordinates ξ^0, …, ξ^{n-1} left with which the manifold can be parametrised on a chart φ.

Such a chart would be the bijective mapping

$$\varphi\colon \begin{pmatrix} x^0 \\ \vdots \\ x^{m-1} \end{pmatrix} \mapsto \begin{pmatrix} \xi^0 \\ \vdots \\ \xi^{n-1} \end{pmatrix} \quad ,$$

or, in a notation more familiar to the physicist:

$$\begin{pmatrix} \xi^0 \\ \vdots \\ \xi^{n-1} \end{pmatrix} = \begin{pmatrix} \xi^0(\mathbf{r}) \\ \vdots \\ \xi^{n-1}(\mathbf{r}) \end{pmatrix} = \boldsymbol{\varphi}(\mathbf{r}) \quad .$$

The ξ^μ are the co-ordinates of those points on the *chart* onto which the points of the *manifold* are mapped, so it is, in fact, the inverse function that is needed to express $\mathbf{r}$ as a function of these free parameters:—

$$\varphi^{-1}\colon \begin{pmatrix} \xi^0 \\ \vdots \\ \xi^{n-1} \end{pmatrix} \mapsto \begin{pmatrix} x^0 \\ \vdots \\ x^{m-1} \end{pmatrix} \quad ,$$

or, shorter:

$$\mathbf{r} = \mathbf{r}(\xi^0, \ldots, \xi^{n-1}) = \boldsymbol{\varphi}^{\mathbf{-1}}(\xi^0, \ldots, \xi^{n-1}) \; .$$

The tangent vector now is simply

$$\mathbf{t} = \frac{\partial \mathbf{r}}{\partial \xi^\mu} \dot{\xi}^\mu \ , \tag{23}$$

as the chain rule demands. This can be reconciled with the notion of manifolds and charts:—

$(\xi^0[t], \ldots, \xi^{n-1}[t])^T = \varphi \circ \gamma$, because the $\xi^\mu(t)$ form the image of the curve γ on the chart φ. So I can rewrite eqn (23) in the form

$$\mathbf{t} = \frac{\partial \boldsymbol{\varphi}^{-1}}{\partial \xi^\mu} \frac{d\,(\varphi \circ \gamma)^\mu}{dt} \quad ,$$

in accord with the idea behind eqs (21) and (22).

Focussing on $m = 4 + 1$, $n = 3 + 1$ (space-times!) and re-parametrising with the arc length, eqn (23) can be rewritten as

$$\mathbf{t} = \frac{\partial \mathbf{r}}{\partial \xi^\mu} u^\mu \ , \tag{24}$$

with the 4velocity $u^\mu = \xi^{\mu\prime}$. This equation shows that the components of the 4velocity coincide with those of the tangent vector of a trajectory, expressed within the basis provided by the vectors $\partial\mathbf{r}/\partial\xi^\mu$, which are the tangent vectors of the parameter lines.

5.3 The Tangent Vector of the Geodesics of Freely Falling Particles in $dS^2 \subset \mathbb{M}^3$

5.3.1 Static Co-Ordinates

The position vector of a point in $\mathbb{M}^3$ be $\mathbf{r} = (x^0, x^1, x^4)^T$. In static co-ordinates, the x^A are expressed by eqs (16):

$$\begin{aligned} x^0(t,r) &= a\,f\ \mathrm{sh}\,\frac{ct}{a}\ , \\ x^1(t,r) &= r\ , \\ x^4(t,r) &= a\,f\ \mathrm{ch}\,\frac{ct}{a}\ , \end{aligned} \tag{16}$$

with

$$f = \sqrt{1 - \frac{r^2}{a^2}} \quad .$$

In the language of differential geometry, eqs (16) represent the mapping φ^{-1} from the chart provided by $\xi^0 = ct$, $\xi^1 = r$ onto the manifold dS^2.

The two basis vectors of the tangential space are

$$\mathbf{e}_0 = \frac{1}{c}\frac{\partial \mathbf{r}}{\partial t} = f \begin{pmatrix} \mathrm{ch}\,\frac{c\,t}{a} \\ 0 \\ \mathrm{sh}\,\frac{c\,t}{a} \end{pmatrix} ,$$

$$\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial r} = \begin{pmatrix} -\frac{r}{a}\,h\,\mathrm{sh}\,\frac{c\,t}{a} \\ 1 \\ -\frac{r}{a}\,h\,\mathrm{ch}\,\frac{c\,t}{a} \end{pmatrix} , \tag{25}$$

$h = 1/f$.

Now the "4velocity" of a body at rest that just follows the expansion of the universe is $(u^\mu) = (h^2, r/a)^T$ in static co-ordinates, see eqn (11), so that the tangent vector of its trajectory is

$$\begin{aligned} \mathbf{t} &= h^2\,\mathbf{e}_0 + \frac{r}{a}\,\mathbf{e}_1 \\ &= \begin{pmatrix} h\,\mathrm{ch}\,\frac{c\,t}{a} - h\,\frac{r^2}{a^2}\;\mathrm{sh}\,\frac{c\,t}{a} \\ \frac{r}{a} \\ h\,\mathrm{sh}\,\frac{c\,t}{a} - h\,\frac{r^2}{a^2}\;\mathrm{ch}\,\frac{c\,t}{a} \end{pmatrix} . \end{aligned} \tag{26}$$

The scalar products $\mathbf{e}_\mu \cdot \mathbf{e}_\nu$ result in the metrical tensor $(g_{\mu\nu}) = \mathrm{diag}(f^2, -h^2)$, where care must be taken because the ambient space is $\mathbb{M}^3$, which means that the scalar product is defined as

$$\mathbf{a} \cdot \mathbf{b} = a^0 b^0 - a^1 b^1 - a^4 b^4 .$$

5.3.2 Planar Co-Ordinates

The components of $\mathbf{r}$ in $dS^2 \subset \mathbb{M}^3$ are now given by the eqs (13):—

$$\begin{aligned} x^0(\bar{t},\chi) &= a\,\mathrm{sh}\,\frac{c\bar{t}}{a} + \frac{1}{2a}\chi^2\,e^{\frac{c\bar{t}}{a}} , \\ x^1(\bar{t},\chi) &= \chi\,e^{\frac{c\bar{t}}{a}} , \\ x^4(\bar{t},\chi) &= a\,\mathrm{ch}\,\frac{c\bar{t}}{a} - \frac{1}{2a}\chi^2\,e^{\frac{c\bar{t}}{a}} . \end{aligned} \tag{13}$$

The two basis vectors that span the tangential space are

$$\bar{\mathbf{e}}_0 \quad = \quad \frac{1}{c}\,\frac{\partial\,\mathbf{r}}{\partial\bar{t}} = \begin{pmatrix} \mathrm{ch}\,\dfrac{c\,\bar{t}}{a} + \dfrac{1}{2}\,\dfrac{\chi^2}{a^2}\;\exp\left(\dfrac{c\,\bar{t}}{a}\right) \\ \dfrac{\chi}{a}\;\exp\left(\dfrac{c\,\bar{t}}{a}\right) \\ \mathrm{sh}\,\dfrac{c\,\bar{t}}{a} - \dfrac{1}{2}\,\dfrac{\chi^2}{a^2}\;\exp\left(\dfrac{c\,\bar{t}}{a}\right) \end{pmatrix} \quad ,$$

$$\bar{\mathbf{e}}_1 \quad = \quad \frac{\partial\,\mathbf{r}}{\partial r} = \begin{pmatrix} \dfrac{\chi}{a} \\ 1 \\ -\dfrac{\chi}{a} \end{pmatrix} e^{\frac{c\,\bar{t}}{a}} \quad . \tag{27}$$

As the “4velocity” of a freely falling observer in these co-ordinates is just $(\bar{u}^{\mu}) = (1,0)^T$, the tangent vector of his trajectory is

$$\mathbf{t} = \bar{\mathbf{e}}_0\ . \tag{28}$$

5.3.3 Switching between Co-Ordinates

Invariance of tangent vector. As both the first equation (26) as well as equation (28) hold for the same tangent vector $\mathbf{t}$, their right-hand sides must likewise be equal. This means that

$$h^2\,\mathbf{e}_0 + \frac{r}{a}\,\mathbf{e}_1 = \bar{\mathbf{e}}_0$$

must obtain after the co-ordinates t, r have been expressed by $\bar{t},\chi$, according to prop. 1.

Transformation of co-ordinates. As

$$\mathbf{e}_{\mu}u^{\mu} = \bar{\mathbf{e}}_{\mu}\bar{u}^{\mu} = \mathbf{t} = \text{inv.}\ ,$$

the basis vectors transform contragrediently to the u^{μ}. This means that

$$\bar{\mathbf{e}}_{\mu} = \mathbf{e}_{\nu}\,\frac{\partial\xi^{\nu}}{\partial\bar{\xi}^{\mu}}$$

transforms as a covariant vector:

$$\begin{aligned} \bar{\mathbf{e}}_0 \quad &= \quad \mathbf{e}_0\,\frac{\partial t}{\partial\bar{t}} + \frac{1}{c}\,\mathbf{e}_1\,\frac{\partial r}{\partial\bar{t}}\ , \\ \bar{\mathbf{e}}_1 \quad &= \quad c\,\mathbf{e}_0\,\frac{\partial t}{\partial\chi} + \mathbf{e}_1\,\frac{\partial r}{\partial\chi}\ . \end{aligned}$$

It can be shown by a straightforward calculation that these three relations hold.

A Derivation of the Four-Acceleration in Robertson-Walker Co-Ordinates

In R. W. co-ordinates the 4acceleration is given by

$$\bar{w}^{\mu} = \frac{d\bar{u}^{\mu}}{ds} + \overline{\Gamma}^{\mu}_{\varrho\sigma}\,\bar{u}^{\varrho}\,\bar{u}^{\sigma} \quad .$$

As just $\bar{u}^0$ and $\bar{u}^1$ are nonzero, the only Christoffel symbols that appear are

$$\begin{aligned} \overline{\Gamma}^{1}_{01} &= \frac{1}{c}\frac{d\ln R}{d\bar{t}} = \frac{1}{a} \quad , \\ \overline{\Gamma}^{0}_{11} &= \frac{1}{c}R\frac{dR}{d\bar{t}} = \frac{e^{\frac{2c\bar{t}}{a}}}{a} \quad , \end{aligned}$$

according to eqn (4).

Now I need

$$\begin{aligned} \frac{d\bar{u}^0}{ds} &= \frac{\partial\bar{u}^0}{\partial\bar{x}^0}\,\bar{u}^0 + \frac{\partial\bar{u}^0}{\partial\chi}\,\bar{u}^1 \\ &= \frac{\chi^2}{a^3}\frac{\exp\left(\frac{2c\bar{t}}{a}\right)}{\left\{1-\frac{\chi^2}{a^2}\exp\left(\frac{2c\bar{t}}{a}\right)\right\}^2} - \frac{\chi^2}{a^3}\frac{\exp\left(\frac{2c\bar{t}}{a}\right)}{\left\{1-\frac{\chi^2}{a^2}\exp\left(\frac{2c\bar{t}}{a}\right)\right\}^2} \\ &= 0 \end{aligned}$$

and

$$\frac{d\bar{u}^1}{ds} = \frac{\partial\bar{u}^1}{\partial\bar{x}^0}\,\bar{u}^0 + \frac{\partial\bar{u}^1}{\partial\chi}\,\bar{u}^1 \quad .$$

According to eqn (9),

$$\bar{u}^1 = -\bar{u}^0\,\frac{\chi}{a} \quad ,$$

so that

$$\begin{aligned} \frac{d\bar{u}^1}{ds} &= -\frac{\chi}{a}\underbrace{\bar{u}^{0\prime}}_{=0} - \bar{u}^0\,\frac{1}{a}\,\bar{u}^1 \\ &= \frac{\chi}{a^2}\frac{1}{1-\frac{\chi^2}{a^2}\exp\left(\frac{2c\bar{t}}{a}\right)} \quad . \end{aligned}$$

Now I can write down

$$\bar{w}^0 = \overline{\Gamma}^{0}_{11}\,(\bar{u}^1)^2 = \frac{\frac{\chi^2}{a^3}\exp\left(\frac{2c\bar{t}}{a}\right)}{1-\frac{\chi^2}{a^2}\exp\left(\frac{2c\bar{t}}{a}\right)}$$

and

$$
\begin{aligned}
\bar{w}^1 &= \frac{d\bar{u}^1}{ds} + 2\,\overline{\Gamma}^{\,1}_{\,01}\,\bar{u}^0\,\bar{u}^1 \\
&= \frac{\chi}{a^2}\,\frac{1}{1-\dfrac{\chi^2}{a^2}\,\exp\left(\dfrac{2c\bar{t}}{a}\right)} + \frac{2}{a}\,\frac{-\dfrac{\chi}{a}}{1-\dfrac{\chi^2}{a^2}\,\exp\left(\dfrac{2c\bar{t}}{a}\right)} \\
&= -\frac{\dfrac{\chi}{a^2}}{1-\dfrac{\chi^2}{a^2}\,\exp\left(\dfrac{2c\bar{t}}{a}\right)} \quad .
\end{aligned}
$$

These components of the 4acceleration are the same as those obtained in eqn (10).